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ON THE INTERPRETATION OF THE MATERNITY FUNCTION AS A PROBABILITY DENSITY

By

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Contents

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SUMMARY

It is noted that the net and gross maternity functions (after division by the appropriate reproduction rates) are often treated as if they were probability densities, and their moments are handled accordingly. This notion is investigated in a probabilistic framework. It turns out that the properties of the moments correspond closely, although not always completely, to their classical, pseudo-probabilistic interpretation in the demographic literature. No meaningful random variable is found, however, which has a probability density proportional to the maternity function, except in the imaginary case of a stationary population.

1. Introduction

A. Let us introduce some terminology and notation, so that we can begin to discuss certain points which we want to raise. Let ℓ be the life table survival function for females, with $\ell(0) = 1$, which means that $\ell(x)$ is the probability that a newly-born girl will survive to age x. Let ϕ be the force of fertility, so that $\phi(x)dx$ is the probability that a woman at age x will have a birth within age x+dx (Hoem, 1969, 1970). We shall also call ϕ the gross maternity function. The function Ψ, defined by

(1) $\quad \Psi(x) = \phi(x)\,\ell(x),$

is known as the net maternity function (Keyfitz, 1968, page 100. Note that Keyfitz uses m for our ϕ, and ϕ for our Ψ.) $\phi(x)$ and $\Psi(x)$ are positive for $\alpha<x<\beta$, and they equal zero elsewhere. The age interval $<\alpha, \beta>$ is the reproductive period.

We let

(2) $\quad R_a = \int_\alpha^\beta x^a \Psi(x)dx$ and $\overline{R}_a = \int_\alpha^\beta x^a \phi(x)dx$, for a = 0, 1, ...

Then R_0 and $\overline{R}_0$ are the usual net and gross reproduction rates, respectively (Keyfitz, 1968, pages 102, 437; Hoem, 1969).

B. The maternity functions will typically look like a left-skewed unimodal probability density, much like a gamma density and some of the beta densities. The maternity functions are not densities themselves, since R_0 and $\overline{R}_0$ seldom equal 1. Some authors, like Tekse (1967), Keyfitz (1968, particularly page 438), and Talwar (1970), treat $\Psi(\cdot)/R_0$ as if this function were a probability density, however, and behave as if

$$\mu = R_1/R_0 \text{ and } \sigma^2 = R_2/R_0 - \mu^2$$

were the mean and variance of a probability distribution. Keyfitz (1968, page 140) calls μ the mean age of childbearing and σ^2 the variance of the age at childbearing, both in the stationary population.

C. It is quite clear that $\Psi(\cdot)/R_0$ is not initially defined as a probability density. Its definition is (1), where ϕ is a force of transition in a Markov chain and ℓ is a transition probability in the same chain (Hoem, 1969). The purpose of the present note is to discuss some interpretations of $\Psi(\cdot)/R_0$ and the moments in (2), and in particular to investigate the interpretations mentioned above. (Of course $\Psi(\cdot)/R_0$ trivially is a density of a constructed random variable like all non-negative real functions with integral 1. We are looking for a meaningful random variable for which it is the density, however.)

In order to facilitate the reading of the paper for those who are not particularly interested in proofs, we give an account of the interpretations in sections 2 to 4 without proofs, and then collect all proofs in Appendices A to C.

2. First interpretation: Age at childbearing as an attribute of the mother

A. Imagine that we follow a group of $\underline{m}$ women throughout the reproductive period and record the ages at which their births occur. Say that the j-th woman has N_j births in all, and the k-th at age X_{kj}. Let $S_j = \Sigma_k X_{kj}$ (interpreted as zero when $N_j = 0$), let

$$\overline{X}_j = S_j/N_j$$

be her mean age at childbearing if $N_j>0$, and let

$$\overline{\overline{X}} = \Sigma_j S_j/\Sigma_j N_j$$

be the corresponding grand mean. As one would expect, this and the other empirical moments that one might compute are closely connected with the R_a in (2). In fact, it is possible to show (see Appendix A) that with probability 1,

$$\text{(3)} \qquad \lim_{m\to\infty} \Sigma_j\Sigma_k X_{kj}^a/\Sigma_j N_j = R_a/R_0 .$$

Thus μ can be interpreted as the (almost certain) limit of the grand mean age $\overline{\overline{X}}$ at childbearing as the size m of the group studied increases without bounds, and the other moments have a similar interpretation.

We shall also show (in Appendix A) that

$$\text{(4)} \qquad E\{\Sigma_k X_{kj}^a\} = R_a,$$

which gives a direct interpretation of the R_a.

B. In many situations one achieves some simplification and one is also able to get further results by conditioning upon the event A that the female survives to the end β of the reproductive period. This is true here also. We can prove (Appendix B) that

$$\text{(5)} \qquad E(\overline{X}_j|N_j>0 \text{ and } A) = \overline{R}_1/\overline{R}_0.$$

The connection among the higher moments is not so simple. <u>In particular</u>, $\overline{R}_2/\overline{R}_0-(\overline{R}_1/\overline{R}_0)^2$ <u>is not the variance of</u> $\overline{X}_j$, <u>given that</u> $N_j>0$ <u>and that the woman survives to age</u> β. In fact, we shall show (in Appendix B) that one rather has

$$\text{(6)} \qquad E(\overline{X}_j^2|N_j>0 \text{ and } A) = \frac{\overline{R}_2}{\overline{R}_0} h(\overline{R}_0)+2\,\frac{\overline{R}_{0,0}}{\overline{R}_0^2}\{1-h(\overline{R}_0)\}.$$

Here

$$\overline{R}_{0,0}= \int_\alpha^\beta \int_x^\beta \phi(x)\phi(y)dydx$$

and

$$h(x) = \int_0^x \frac{e^y-1}{y}\, dy/(e^x-1).$$

If $\overline{X}_j$ had the conditional density $\phi(\cdot)/\overline{R}_0$, given A and $N_j>0$, then $E(\overline{X}_j^2|N_j>0$ and A) would equal $\overline{R}_2/\overline{R}_0$. (6) shows that this is not the case, so $\phi(\cdot)/\overline{R}_0$ is not the conditional density of $\overline{X}_j$.

3. Second interpretation: Age at childbearing as an attribute of the child

In section 2, we imagined that each woman was followed over the reproductive period. Nathan Keyfitz has suggested in a private communication that we might use a different observational scheme. We could consider a population at some moment, called time zero, and could follow the entire population over some suitable period. Whenever a birth occurred, we could record the age of the mother, which would then become an attribute of the child. Characteristics of the ages at childbearing collected in this fashion might give us an interpretation of the sort suggested in section 1B.

In order to investigate this possibility, we introduce an initial age distribution with probability density $p(\cdot)$ in the female population. Thus, if at time zero we pick a female at random and observe her age X, then

$$P\{x<X<x+dx\} = p(x)dx.$$

Let $0\leq t<\alpha$, so that no birth observed at time t can be due to a woman born during the study period. Let the population be closed, and let B_t be the event that we observe a birth at time t. (The possibility of observing two or more births simultaneously can be disregarded.) Given B_t, let Y be the age of the mother having this birth. Then, for $\alpha<y<\beta$, the conditional density of Y is given by

$$f_t(y)dy = P\{y<Y<y+dy|B_t\} = c(t)p(y-t)\frac{\ell(y)}{\ell(y-t)}\,\phi(y)dy, \tag{7}$$

where

$$c(t) = 1/\int_\alpha^\beta p(x-t)\,\frac{\ell(x)}{\ell(x-t)}\,\phi(x)dx. \tag{8}$$

Thus the density $f_t(\cdot)$ of Y, given B_t, generally depends very much both on t and on the initial age distribution $p(\cdot)$, as one might expect. If a particular age group is unusually scarce (or plentiful) at time zero, then the effect of this on the age distribution of mothers in subsequent periods will be felt as long as the corresponding women are having births.

Assume, however, that the initial age distribution is stable, i.e. that

$$p(x) = e^{-rx}\ell(x)/\int_0^\omega e^{-ry}\ell(y)dy.$$

Then

$$f_t(y) = \frac{e^{-ry}\phi(y)\ell(y)}{\int_\alpha^\beta e^{-rx}\phi(x)\ell(x)dx},$$

so that the density is independent of t. (Compare Keyfitz, 1968, page 126,(5.6.8).)

If, in addition, $r=0$, so that the population is stationary, we get

$$f_t(y) = \Psi(y)/R_0.$$

In this sense, therefore, $\Psi(\cdot)/R_0$ is the density of the age at childbearing in the stationary population, and the interpretations of μ and σ^2 given in section 1B are correct.

4. Third interpretation: A construct

We shall give a final interpretation, where again $\Psi(\cdot)/R_0$ is the probability density of a nontrivial random variable. This interpretation has the characteristics of a construct, and is therefore of much less interest than the previous two. It is published here in the hope that some reader may find a useful interpretation.

Let us return to the situation in section 2, where we follow a woman through the reproductive period. We drop the subscript j. Let K be an integer random variable which, when $N = n>0$, is independent of $X_1,\ldots,X_n$ and has the distribution

$$P\{K=k|N=n\} = \frac{1}{n} \quad \text{for } k=1,2,\ldots,n. \tag{9}$$

Let $Y=X_K$. Then the probability density of Y, given $N>0$, is $\Psi(\cdot)/R_0$, as we shall show in Appendix C.

I have not been able to find a random variable K which both has these properties and also has a reasonable intuitive content.

Appendices

A. Proof of (3) and (4): Let $A_k(x,t)$ be the probability that a female of age x will have k births in the age interval $[x,x+t\rangle$, and let $P_k(x,t)$ be the probability that she has these births and also survives to age x+t. Let

$$\Phi(x,t) = \int_0^t \phi(x+\tau)d\tau,$$

and let

$$(10)\qquad \overline{P}_k(x,t) = \frac{1}{k!}[\Phi(x,t)]^k \exp\{-\Phi(x,t)\}.$$

Then (Hoem, 1969, (2.5))

$$P_k(x,t) = \overline{P}_k(x,t)\ell(x+t)/\ell(x).$$

We introduce

$$(11)\qquad a_n = P\{N_j=n\} = A_n(0,\beta),$$

and get, for n>0,

$$(12)\qquad P\{x<X_{kj}<x+dx|N_j=n\} = a_n^{-1}P_{k-1}(0,x)\phi(x)A_{n-k}(x,\beta-x)dx.$$

Thus

$$E\{\sum_{k=1}^n X_{kj}^a|N_j=n\} = a_n^{-1}\int_\alpha^\beta x^a\phi(x)\ell(x)\sum_{k=1}^n \overline{P}_{k-1}(0,x)A_{n-k}(x,\beta-x)dx.$$

Since

$$\sum_{n=1}^\infty \sum_{k=1}^n \overline{P}_{k-1}(0,x)A_{n-k}(x,\beta-x) = \sum_{k=1}^\infty \overline{P}_{k-1}(0,x)\sum_{n=k}^\infty A_{n-k}(x,\beta-x)=1,$$

we get (4). On the assumption of independence between the reproductive histories of the women, we then get (3) by the standard formula $R_0=EN_j$ (use (4) with $a = 0$) and the strong law of large numbers.

B. Proof of (5) and (6): In appendix B we drop the subscript j and assume that $\ell(\beta)=1$. (One may show (Hoem, 1969) that the latter assumption corresponds to conditioning on the event A.) By (10), (11), and (12), we get

$$a_n = \frac{\overline{R}_0^n}{n!} e^{-\overline{R}_0}$$

and, for $1\leq k\leq n, \alpha<x<\beta$,

$$(13)\qquad P\{x<X_k<x+dx|N=n\}=n\overline{R}_0^{-n}\phi(x)\binom{n-1}{k-1}\Phi(0,x)^{k-1}\Phi(x,\beta-x)^{n-k}dx.$$

For $1\leq k<j\leq n, \alpha<x<y<\beta$, we get

$$P\{x<X_k<x+dx, y<X_j<y+dy|N=n\}=$$

$$=n(n-1)\overline{R}_0^{-n}\phi(x)\phi(y)\frac{(n-2)!}{(k-1)!(j-k-1)!(n-j)!}\Phi(0,x)^{k-1}\Phi(x,y-x)^{j-k-1}\Phi(y,\beta-y)^{n-j}dxdy$$

Thus

(14) $$E\{\sum_{k=1}^{n} X_k^a | N=n\} = n\bar{R}_a/\bar{R}_0$$

and

(15) $$E\{\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} X_k X_j | N=n\} = n(n-1)\bar{R}_{0,0}/\bar{R}_0^2 .$$

(5) follows directly from (14). To get (6), we proceed as follows:

$$E\{\bar{X}^2 | N>0\} = \sum_{n=1}^{\infty} n^{-2} E\{\sum_{k=1}^{n} X_k^2 + 2\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} X_k X_j | N=n\} a_n/(1-a_0)$$

$$= \sum_{n=1}^{\infty} n^{-1}\{\bar{R}_2/\bar{R}_0 + 2(n-1)\bar{R}_{0,0}/\bar{R}_0^2\} a_n/(1-a_0).$$

Let

$$g(x) = \sum_{n=1}^{\infty} n^{-1} x^n/n! = \int_0^x y^{-1}(e^y-1)dy.$$

(The integral formula is proved by differentiation.)

Then

$$\sum_{n=1}^{\infty} n^{-1} a_n = e^{-\bar{R}_0} g(\bar{R}_0)$$

and consequently

$$E\{\bar{X}^2 | N>0\} = (\bar{R}_2/\bar{R}_0 - 2\bar{R}_{0,0}/\bar{R}_0^2) e^{-\bar{R}_0} g(\bar{R}_0)/(1-a_0) + 2\bar{R}_{0,0}/\bar{R}_0^2.$$

(6) then follows.

C. Proof of the statement below (9): By (13),

$$P\{y<Y<y+dy | N=n\} = \sum_{k=1}^{n} P\{y<X_k<y+dy, K=k | N=n\} =$$

$$= \sum_{k=1}^{n} \bar{R}_0^{-n} \phi(y) \binom{n-1}{k-1} \Phi(0,y)^{k-1} \Phi(y,\beta-y)^{n-k} dy = \phi(y)dy/\bar{R}_0.$$

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