Arbeidsnotater

STATISTISK SENTRALBYRÅ

WORKING PAPERS FROM THE CENTRAL BUREAU OF STATISTICS OF NORWAY

IO 70/23 Oslo, 15. desember 1970

MODIS IV
THE BASIC FRAMEWORK OF AN INPUT-OUTPUT PLANNING MODEL, WITH A COMMODITY-ACTIVITY-SECTOR APPROACH.*

By

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CONTENTS

* A paper to be presented to the Fifth International Conference on Input-Output Techniques in Geneva, January 11-15, 1971. The paper is an outcome of a series of discussions between the authors and other members of the research staff of the Central Bureau of Statistics. The authors recognize the impossibility of distinguishing their own contributions to the content of this paper from those of their senior colleagues Arne Amundsen and Per Sevaldson to whom they wish to express their sincere thanks also for valuable support in the drafting of the paper.

1. INTRODUCTION

MODIS IV will be the fourth version in a series of macro-economic models constructed and used by the Central Bureau of Statistics of Norway.[1] Like its predecessors MODIS IV will combine a disaggregated input-output framework with a number of additional relations and auxiliary assumptions. In the earlier versions the input-output framework was a square matrix of coefficients relating sector inputs and sector outputs in a way closely related to the original Leontief scheme.

The other main elements of the earlier versions were price relations based on input-output cost calculations, consumption relations determining the volume and commodity distribution of private consumption, and tax and income relations. The number of production sectors in the predecessors was about 150.

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[1] The earlier versions were named MODIS I, II and III.

MODIS I which was in operation from 1960 to 1965 is outlined and discussed in:

Per Sevaldson: An Interindustry Model of Production and Consumption in Norway. Income and Wealth. Series X. London 1964.

MODIS II which was in operation from 1965 to 1967 is presented in:

Per Sevaldson: MODIS II. A macro-economic model for short-term analysis and planning. Artikler from the Central Bureau of Statistics, no. 23, 1968.

MODIS III which is the current version is described and discussed in:

Olav Bjerkholt: A precise description of the system of equations of the economic model MODIS III. Artikler from the Central Bureau of Statistics, no. 24, 1968.

Per Sevaldson: Data Sources and User Operations of MODIS, a Macro-economic Model for Short-term Planning. Paper presented at UN First Seminar on Mathematical Methods and Computer Techniques in Bulgaria 1970.

The MODIS models have been used to a great extent by the Ministry of Finance in the preparation of one-year plans (National Budgets) and the detailed specifications of the models have been strongly influenced by the needs of this purpose.

The extreme simplicity of the original Leontief model, including the duality between the price relations and the quantity relations, was to a great extent preserved in the earlier MODIS versions. The benefits of this simplicity have been quite substantial for practical applications of the models. It has been found, however, that the simple input-output model does not take full advantage of the available input-output data base.

The activity formulation of the input-output model is meant to represent a conceptual as well as an empirical improvement over the traditional input-output framework. The activity analysis framework will distinguish explicitly between three different aspects of commodity production in sectors, namely, the way of producing which is a technological concept, the result of production which is a commodity concept, and the localization of production in establishments which is an organisational concept. The sector concept of the traditional input-output model implicitly identifies these three aspects.

The input-output framework of MODIS IV will, accordingly, differ from the traditional square matrix of input-output coefficients. The framework adopted is more related to that of the general linear activity analysis.[2)] The main concepts of the model framework are introduced and defined in section 2.

In section 3 and 4 the concepts of the activity analysis framework are given empirical content, and it is described how the coefficients of the activity framework are to be determined from the input-output data base of the national accounts. Readers uninterested in the empirical content of the concepts of this framework may skip section 3 and 4.

The activity analysis framework is well suited for embedding into wider model frameworks. Some possibilities of combining the input-output framework with other relations are outlined in section 5.

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2) See e.g. Koopmans, ed.: Activity Analysis of Production and Allocation. London 1951.

2. BASIC CONCEPTS OF THE ACTIVITY MODEL

The two central concepts of the Activity model are Commodity and Activity. By Commodity is meant a grouping of goods and services. The Commodities include as proper aggregates all commodities of the Bruxelles nomenclature. Other Commodities are grouping of services. The number of Commodities is n_X.

By Activity is meant a process which transforms a set of input Commodities in fixed proportions into a set of output Commodities in fixed proportions. The scale of an Activity is a continuous non-negative variable called the Activity level. The fixed proportions of input and output are preserved under scale variations. The input set or the output set of Commodities for an Activity may be empty. Activities with an empty set of input as well as output Commodities are not allowed. The number of Activities is n_A.

The Activities are given as two matrices, Λ^- and Λ^+, containing the fixed proportions of input and output Commodities respectively. We assume that inputs and outputs of Commodities are valued in a set of fixed prices. The total input to or output from an Activity can on this assumption be measured in currency units.

(2.1) Λ^- = "Activity input matrix", Dim $\Lambda^- = (n_X, n_A)$

The element on row i and column j of Λ^- is equal to the ratio between input of Commodity i in Activity j and the total input of Activity j.

(2.2) Λ^+ = "Activity output matrix", Dim $\Lambda^+ = (n_X, n_A)$

The element on row i and column j of Λ^+ is equal to the ratio between output of Commodity i in Activity j and the total output of Activity j.

(2.3) A^- = "Activity inputs", Dim $A^- = n_A$

The elements of A^- are equal to total input of the n_A Activities.

(2.4) A^+ = "Activity outputs", Dim $A^+ = n_A$

The elements of A^+ are equal to total output of the n_A Activities.

(2.5) X^- = "Commodity inputs", Dim X^- = n_X

The elements of X^- are equal to the sum of input to all Activities of the n_X Commodities.

(2.6) X^+ = "Commodity outputs", Dim X^+ = n_X

The elements of X^+ are equal to the sum of output from all Activities of the n_X Commodities.

From the definitions (2.1) - (2.6) two immediate relations follow.

(2.7) $X^- = \Lambda^- A^-$

(2.8) $X^+ = \Lambda^+ A^+$

The Activities comprise all sources of supply and all uses of the various Commodities recognized by the national accounts with the exceptions mentioned after (2.13) below. The Activities can be subdivided accordingly as in the following list.

I. Import Activities (B)

II. Production Activities (P)

III. Final demand Activities

(i). Private consumption Activities (C)
(ii). Public consumption Activities (G)
(iii). Gross investment Activities (I)
(iv). Export Activities (E)

(The capital letters on each line will later be used as subscripts by subdivision of Activities).

The Activities in group I have zero input columns while Activities in group III have zero output columns.

The Activity level is equal to the absolute value of the difference between the total output of Commodities from and input of Commodities to an Activity.

(2.9) $A = |A^+ - A^-|$ = "Activity levels", Dim A = n_X

It follows that the Activity level of an import Activity is equal to total output, the Activity level of a final demand Activity is equal to total input, and the Activity level of a production Activity is equal to net input which will be denoted beside "Activity level" as "gross product of the Activity".

Activity levels are obviously non-negative.

For production Activities nothing has been said so far about the proportions between total input and total output of the Activities. Formally, these proportions are determined by two vectors of productivity coefficients.

(2.10) η^{-} = "input productivity coefficients", Dim $\eta^{-} = n_A$

The elements of η^{-} are equal to the ratios between the total input and the Activity level of each Activity.

(2.11) η^{+} = " output productivity coefficients", Dim $\eta^{+} = n_A$

The elements of η^{+} are equal to the ratios between the total output and the Activity level of each Activity.

By definition the elements of η^{-} are equal to zero for import Activities and equal to one for final demand Activities. For the elements of η^{+} the opposite is the case. The function of the productivity coefficients is to normalize the input and output columns of the production Activities.

(2.12) $\Lambda = \Lambda^{+}\hat{\eta}^{+} - \Lambda^{-}\hat{\eta}^{-}$ = "Activity matrix", Dim $\Lambda = (n_X, n_A)$

The proper interpretation of Λ is net output of the various Commodities per unit of Activity levels.

(2.13) $X = X^{+}-X^{-}$ = "Commodity surplus", Dim $X = n_X$

The Commodity surplus vector is the net surplus (which may be negative) from all Activities of the n_X Commodities. The neutral term "net surplus" has been chosen because this vector may have different interpretations depending on the exact specification of the model. The net surplus may include, for instance, an unexplained residual, net addition to stocks, or other final demand items not accounted for by Activities. It may even be assumed to be zero by definition.

The basic equation of the Activity model is the following.

(2.14) $X = \Lambda A$

This equation follows rather trivially from the definitions above. It comprises n_X relations between (n_X+n_A) variables (assuming the matrix Λ to be a known matrix). We shall discuss at some length in section 5 various ways of "closing" a model including equation (2.14), that is various ways of adding to (2.14) a sufficient set of additional relations and assumptions of exogenous variables to determine all elements of A and X. The discussion will be extended to include relations between price indices of Commodities and Activity levels and interrelations between price and quantity variables. In the rest of this chapter we shall extend our list of variables.

The Activity matrix and the vector of Activity levels can be subdivided by type of Activity and the equation (2.14) can be rewritten as

$$(2.15) \quad X = \Lambda_B A_B + \Lambda_P A_P + \Lambda_C A_C + \Lambda_G A_G + \Lambda_I A_I + \Lambda_E A_E$$

The import Activities given as the columns of Λ_B are typically columns with only one non-zero element. For each imported Commodity there corresponds at least one Activity with a unit element on the row of the Commodity in question. Several import Activities for the same Commodity may be introduced to distinguish between foreign markets. The elements of A_B are equal to the imported quantities of the various Commodities.

The production Activities are related to the columns of the traditional input-output matrix. The fixed ratios between input Commodities are maintained. The production Activities differ, however, from the columns of the traditional input-output matrix in three respects. First, multiple output of Commodities from a single Activity are allowed. Secondly, the same Commodity may be produced in different Activities, and thirdly, the Activity coefficients are normalized by gross product, not gross production. The Activity levels A_P are equal to gross product of the Activities.

The private consumption Activities, Λ_C, are typically Activities with only one input Commodity aside from trade margins. The same is true for the export Activities and to a certain extent for the public consumption Activities and the gross investment Activities. A_C includes all private consumption items. A_G and A_I likewise include all public consumption and gross investment by type of public consumption and type of capital, respectively. A_E will include all or a great part of total export. Minor export Commodities may possibly be included in the Commodity surplus vector.

Each Activity belongs to one and only one Sector. The import Activities are grouped together in an import Sector and similarly for the other subdivisions of Activities except production Activities. The production Activities are grouped in a number of production Sectors. The production Sector comprises a number of production Activities. The number of Sectors is n_S.

Input and output of Sectors can be aggregated from Activity inputs and outputs. The aggregations are performed by an aggregation matrix.

(2.16) Σ = "aggregation matrix", Dim $\Sigma = (n_S, n_A)$

(2.17) S^- = "Sector inputs", Dim $S^- = n_S$

(2.18) S^+ = "Sector outputs", Dim $S^+ = n_S$

The aggregation from Activities to Sectors can then be written as:

(2.19) $S^- = \Sigma A^-$

(2.20) $S^+ = \Sigma A^+$

(The matrix Σ has one unit element in each column and zeros elsewhere).

The Sectors can be subdivided like the Activities. We shall apply the subscript notations for subdivision introduced in (2.15) for Sectors as well as for the aggregation matrix, e.g.

(2.21) $S_P^+ = \Sigma_P A^+$, Dim $S_P^+ = n_{SP}$

Where n_{SP} = the number of production Sectors.

The "Sector levels" can likewise be introduced as

(2.22) $S = \Sigma A$, Dim $S = n_S$

The subvectors of S will be referred to as follows

S_B = "total imports"
S_P = "gross product in Sectors"
S_C = "total private consumption"
S_G = "total public consumption"
S_E = "total exports"

3. CLASSIFICATION OF COMMODITIES AND PRODUCTION SECTORS

In this section we shall discuss the concepts of Commodity and production Sector more thoroughly by presenting the classification which will be used in the model. Commodities and sectors in MODIS IV will be closely related to corresponding concepts in the Norwegian national accounts. For that reason we shall first give a short description of the national accounts of real transactions.

3.1 Commodities and production sectors in the national accounts

The Norwegian national accounts are under revision. The new version will, with some modifications, follow the principles laid down in the new System of National Accounts.[1)] The central part of the national accounts of real transactions will be two matrices, one for the deliveries of commodities <u>to</u> sectors (a sector input table) and one for the deliveries of commodities <u>from</u> sectors (a sector output table). This has in fact for a long time been the structure of the real transactions in the Norwegian national accounts.

In the national accounts there are specified about 1 500 different micro commodities, each being a proper aggregate of commodities in the Bruxelles nomenclature, and nearly 150 production sectors, each being a proper aggregate of establishments. The aggregation of establishments to sectors is based on the Norwegian version of the International Standard Industrial Classification of All Economic Activities. In ISIC the micro commodities are grouped in such a way that each group consists of "similar" micro commodities, and all establishments whose characteristic commodity[2)] belongs to the same ISIC commodity group are aggregated to one ISIC-sector. The composition of output of the establishments will change over time but each establishment will, in principle, be reclassified when the characteristic commodity change. The ISIC-sectors are the starting point for the aggregation to the production sectors of the national accounts.

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1) See A System of National Accounts, Department of Economic and Social Affairs, United Nations, New York, 1968.

2) The characteristic commodity of an establishment is defined as the commodity which has the largest share of the value of production of the establishment. Corresponding definitions will be used for the characteristic commodity of an Activity and of a Sector.

3.2 Commodities and production Sectors in MODIS IV

The sector concept in the model is identical with the sector concept in the national accounts. This means that MODIS IV will get nearly 150 Production Sectors which is about the same number as in MODIS III.

The micro commodities in the national accounts are on the other hand too disaggregated to serve as Commodities in the model. In principle, only micro commodities which either are produced for technical reasons in fixed proportions or have the same input structure and are completely substitutable in the demand should be aggregated. This will give homogenous Commodities. But in addition to that such information in general is not available, this procedure, strictly followed, will result in a too large number of Commodities.

Instead we shall mainly base the aggregation of micro commodities to Commodities on the so-called "principle of main producer". This means that all micro commodities with the same Sector as the main producer will form one Commodity. The procedure will give a proper aggregation of micro commodities to Commodities because each micro commodity will be part of one and only one Commodity. The main producer of quite a few micro commodities will change over time because the composition of the production of the individual establishments change. In practice this process will be rather slow because the establishments, as mentioned, will be reclassified when the characteristic commodity change. It will therefore not be necessary to reclassify the Commodities on the basis of the principle of main producer more than say every fourth of fifth year.[3)]

The result of the procedure of classifying Commodities as outlined above, given that ISIC-sectors are used as the starting point, is completely dependent on the aggregation of ISIC-sectors to production Sectors in the national accounts. The revision of national accounts which now takes place also includes a revision of the sector classification and the objective of getting as homogenous Commodities as possible in MODIS IV will be decisive for this revision.

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3) If we do not change the content of each Commodity from time to time we must increase the number of production Activities over time if the model shall give an adequate description of the production structure of the economy. It is therefore necessary to reclassify the Commodities if we want to keep the size of the model relatively stable. For the classification of production Activities, see section 4.

The principle of main producer will result in the same number of Commodities as production Sectors, given that each production Sector is the main producer of at least one micro commodity. But the classification of Commodities must also take account of the need for an adequate classification of imports and final demand as well as a reasonable representation of the structure of the indirect tax system. This will lead us to subdivide some of the commodities defined according to the principle of main producer. In MODIS IV there will consequently be more Commodities than production Sectors. The number of Commodities will be about 175.

4. THE PRODUCTION ACTIVITY MATRIX (Λ_P)

In this section we shall present an estimation procedure of the elements of the production Activity matrix, denoted by the elements of $\Lambda_P = \Lambda_P^+ \hat{\eta}_P^+ - \Lambda_P^- \hat{\eta}_P^-$, for the base year of the model.[1)]

The Activity matrix is defined from the Activity input and output tables. The Activity input table gives the input of Commodities to Activities and the Activity output table gives the output of Commodities from Activities.

(4.1) W_{AP}^- = "Activity input table", Dim $W_{AP}^- = (n_X, n_{AP})$

The element on row i and column j of W_{AP}^- is the amount of input of Commodity i in Activity j.

(4.2) W_{AP}^+ = "Activity output table", Dim $W_{AP}^+ = (n_X, n_{AP})$

The element on row i and column j of W_{AP}^+ is the amount of output of Commodity i from Activity j.

The elements of the Activity inputs, A_P^-, which are equal to total input of the Activities, and the elements of the Activity outputs, A_P^+, which are equal to total output of the Activities, are simply found as the column sums of the elements of W_{AP}^- and of the elements of W_{AP}^+, respectively.

The elements of the Activity input matrix Λ_P^- are equal to the ratios between the inputs of Commodities and the total input of each Activity.

(4.3) $\Lambda_P^- = W_{AP}^- (\hat{A}_P^-)^{-1}$

The elements of the Activity output matrix Λ_P^+ are equal to the ratios between the outputs of Commodities and the total output of each Activity.

1) In this section we shall for the sake of simplicity in general use the terms Activity and Sector for production Activity and production Sector respectively, because other types of Activities and Sectors will not be discussed.

(4.4) $\Lambda_P^+ = W_{AP}^+ (\hat{A}_P^+)^{-1}$

The elements of the input productivity coefficient vektor η_P^- are equal to the ratio between the element of A_P^- and the element of $A = |A_P^+ - A_P^-|$ for each Activity.

The elements of the output productivity coefficient vektor η_P^+ are equal to the ratio between the element of A_P^+ and the element of $A = |A_P^+ - A_P^-|$ for each Activity.

In general, at the level of aggregation of the model, observations of inputs and outputs of Commodities for Activities (the elements of W_{AP}^- and W_{AP}^+) are not directly available in existing statistical data. In the framework presented here the data base for the estimation of the elements of the Activity input and output tables will mainly be the Sector input and Sector output tables in the new version of the Norwegian national accounts (see section 3.1.) The estimation procedure will , however, allow for use of more direct information, such as engineering data etc., whenever available.

(4.5) W_{SP}^- = "Sector input table", Dim $W_{SP}^- = (n_X, n_{SP})$

The element on row i and column j of W_{SP}^- is the amount of input of Commodity i in Sector j.

(4.6) W_{SP}^+ = "Sector output table", Dim $W_{SP}^+ = (n_X, n_{SP})$

The element on row i and column j of W_{SP}^+ is the amount of output of Commodity i from Sector j.

Each Activity belongs to one and only one Sector. It follows that we may aggregate the columns of W_{AP}^- to W_{SP}^- and the columns of W_{AP}^+ to W_{SP}^+ by means of the aggregation matrix Σ_P, Dim $\Sigma_P = (n_{SP}, n_{AP})$

(4.7) $W_{SP}^- = W_{AP}^- \Sigma_P'$

(4.8) $W_{SP}^+ = W_{AP}^+ \Sigma_P'$

4.1 The production Activity output table $\overset{+}{W}_{A_P}$.

As mentioned, the data base for the determination of the elements of $\overset{+}{W}_{A_P}$ for the base year of the model is the set of observed elements of $\overset{+}{W}_{S_P}$ for the same year. The transition from the Sector output table $\overset{+}{W}_{S_P}$ to the Activity output table $\overset{+}{W}_{A_P}$ is done directly by distributing the outputs of Commodities from each Sector between the Activities in the Sector by help of a set of "Activity classification matrices".

(4.9) $\Gamma_{\cdot q}(q=1, \ldots, n_{S_P})$ = "Activity classification matrix",

$$\text{Dim } \Gamma_q = (n_X, n_{A_q})$$

n_{A_q} is the number of Activities in Sector q.

The element on row i and column j of Γ_q denotes the ratio between output of Commodity i in the j'th Activity of Sector q and the total output of Commodity i in Sector q.

(4.10) $\overset{+}{W}_q(q=1, \ldots, n_{S_P})$ = "Sector q output vector",

$$\text{Dim } \overset{+}{W}_q = n_X$$

The element on row i of $\overset{+}{W}_q$ is the amount of output of Commodity i in Sector q. $\overset{+}{W}_q$ is identical with column q in $\overset{+}{W}_{S_P}$.

By combining (4.9) and (4.10) the elements of $\overset{+}{W}_{A_P}$ are determined.

(4.11) $$\overset{+}{W}_{A_P} = \{\hat{\overset{+}{W}}_1 \Gamma_1, \ldots, \hat{\overset{+}{W}}_{n_{S_P}} \Gamma_{n_{S_P}}\}$$

MODIS IV will, like its predecessors, be used by the Norwegian government several times every year for regular planning purposes. It will therefore be advantageous to establish a set of Activities fitted to this use of the model. This set of Activities, which we shall call the "Ordinary Set", will contain about 250 Production Activities.

The version of $\Gamma_q(q=1, \ldots, n_{S_P})$ that will be used to classify the Ordinary Set of Production Activities will have only unit and zero elements. This means that all production of each Commodity in each Sector is allocated to the same Activity. In the Ordinary Set it will consequently be possible to identify each Activity by stating the characteristic Commodity and the Sector it belongs to, because a Commodity can only be the characteristic Commodity of one Activity in each Sector.

In general, we specify an Activity for each of the Commodities which a Sector produces on a significant scale. The columns of such Activities in the matrices $\Gamma_q (q=1, . . . , n_{SP})$ will consequently contain just one unit element each, namely for the Commodity produced in the Activity, and the other elements will be zeros.

For quite a few Activities we will assume multiple output. In principle, two or more Commodities shall only be allocated to the same Activity if for technical or other reasons it is not possible to vary the ratios between the output quantities of these Commodities. But due to the relatively high level of aggregation in MODIS IV (about 150 production Sectors and 175 Commodities) as well as to the general principle used in the Commodity classification (the principle of main producer) there will be very few Activities with multiple output owing to the technique of production.

On the other hand, in many Sectors there will be quite a few Commodities produced on a too small scale to deserve specification in independent Activities, although we shall go rather far in specifying Activities in the Ordinary Set. Within each Sector such small items will usually be allocated to the Activity which have the same characteristic Commodity as the Sector itself. This procedure will not change the content of these Activities very much because the production of the characteristic Commodity generally will be completely dominating in these Activities.

For an Activity with multiple output the columns in the matrix $\Gamma_q (q=1, . . . , n_{SP})$ will contain a unit element for each Commodity produced by this Activity.

4.2 The production Activity input table $\bar{W}_{AP}$.

The data base for the determination of the elements of $\bar{W}_{AP}$ for the base year of the model is the set of observed elements of $\bar{W}_{SP}$ for the same year. The problem is to distribute the inputs of Commodities in each Sector between the Activities in the Sector. The connection between $\bar{W}_{A_P}$ and $\bar{W}_{S_P}$ is (4.7). In general, there are n_{A_P} different columns in $\bar{W}_{A_P}$ and n_{S_P} independent equations between these columns in (4.7). This means that the matrix $\bar{W}_{A_P}$ at most will have $n_{A_P} - n_{S_P}$ linearly independent columns if the elements of the columns are to be estimated under the restriction (4.7), with $\bar{W}_{S_P}$ and Σ_P given.

We make the system determinate by introducing a dummy matrix T_θ, Dim $T_\theta = (n_X , n_{S_P})$, and by imposing n_{A_P} additional restrictions in the form of n_{A_P} linear relationships between the n_{A_P} columns of $\bar{W}_{A_P}$ and the n_{S_P} columns of T_θ. The linear relations may also include constant terms. This opens for the possibility directly to utilize exogenous informations, such as engineering data etc.

$$(4.12) \quad \bar{W}_{A_P} = T_\theta \, \theta + T_0$$

$$(4.13) \quad \theta = \text{"linear restriction matrix"}, \quad \text{Dim } \theta = (n_{S_P}, n_{A_P})$$

The element on row i and column j of θ denotes the proportion of each element of column i in T_θ which enters into each element of column j in $\bar{W}_{A_P}$.

The subscript of T_θ indicates that the content of T_θ is dependent on the content of the linear restriction matrix θ.

$$(4.14) \quad T_0 = \text{"constant term matrix"}, \quad \text{Dim } T_0 = (n_X, n_{A_P})$$

The element on row i and column j of T_0 denotes the exogenous constant which enters into the element on row i and column j of $\bar{W}_{A_P}$.

With $\bar{W}_{S_P}$, Σ_P, θ and T_0 given (4.7) and (4.12) impose $n_{S_P} + n_{A_P}$ restriction on the $n_{A_P} + n_{S_P}$ columns of $\bar{W}_{A_P}$ and T_θ and the elements of these columns may be found if the system has a solution.

To estimate the elements of $\bar{W}_{A_P}$ we combine (4.7) and (4.12).

$$(4.15) \quad \bar{W}_{S_P} = (T_\theta \, \theta + T_0) \, \Sigma_P'$$

If $\theta \cdot \Sigma'_P$ is nonsingular, which depends on the formulation of the dependence between $\bar{W}_{A_P}$ and T_θ in (4.12), we get

$$(4.16) \quad T_\theta = (\bar{W}_{S_P} - T_0 \Sigma'_P)(\theta \Sigma'_P)^{-1}$$

The elements of $\bar{W}_{A_P}$ are now easily found by combining (4.12) and (4.16).

$$(4.17) \quad \bar{W}_{A_P} = (\bar{W}_{S_P} - T_0 \Sigma'_P)(\theta \Sigma'_P)^{-1} \theta + T_0$$

$$= \bar{W}_{S_P}(\theta\Sigma'_P)^{-1}\theta + T_0(I - \Sigma'_P(\theta\Sigma'_P)^{-1}\theta)$$

As a special case of the general formulation of the dependence between $\bar{W}_{A_P}$ and T_θ in (4.12) we may assume that the n_{A_P} Activities in $\bar{W}_{A_P}$ may be grouped into only n_{S_P} different groups in such a way that all Activities <u>within each group</u> have the same input coefficients (proportional columns in $\bar{W}_{A_P}$ and identical columns in $\bar{\Lambda}_P$). This means that there will be only one element in each column of θ and all Activities with elements different from zero on the same row of θ will belong to the same group. All elements of T_0 will be zero. It also follows that the ratios between the elements different from zero on each row of θ will be the same as the ratios between the total input of the Activities in the group. The elements of $\bar{A}_P$ are, however, unknown as long as the elements of $\bar{W}_{A_P}$ are unknown. A way of solving this problem will be to assume that the ratio between the total input and total output is the same for all Activities <u>within each group</u>. The ratios between the elements on each row of θ will then be the same as the ratios between the total output of the Activities in the group.

In the special case discussed here, the elements of each column of T_θ are relative to the choice of the elements of the corresponding rows of θ. If we choose to let the elements different from zero on each row of θ be the total inputs of the corresponding Activities, which is a convenient normalisation, the elements of the columns of T_θ may be interpreted as Activity input coefficients.

The estimation of the Activity input matrix of the production Activities will, at least in the initial stages of implementing the model, normally be based on the assumptions made for the special case just discussed. The problem is to group the n_{A_P} Activities in n_{S_P} different groups in such a way that it is reasonable to assume that all Activities within each group have the same input coefficients (identical columns in $\bar{\Lambda}_P$).

In the Ordinary Set of production Activities we shall distinguish between two main categories of groups:

(i) Groups with more than one Activity and where all Activities within each group have the same characteristic Commodity. For the Activities within each of these groups we can say we are assuming a characteristic Commodity technology because they are assumed to have the same input structure whichever Sector they belong to

(ii) Groups with one or more Activities where all Activities within each group belong to the same Sector. For the Activities within each of these groups we can say we are assuming a Sector technology.

Within the framework outlined above for the Ordinary Set of Production Activities it is relatively easy to avoid the possibility that (4.7) and (4.12) will form either an overdetermined or an underdetermined system of equations. However, some care is needed. As an example we shall mention that we will get an underdetermined system of equations if there are two or more groups of category (ii) in one or more Sectors.

5. EXTENSIONS AND USE OF THE ACTIVITY ANALYSIS FRAMEWORK

In this section some possibilities of building a macro-economic model around the Activity analysis framework presented in the preceding sections will be examined. The basic equation of this framework is

$$(5.1) \quad \Lambda A = X$$

The elements of A are, according to specifications given in section 2, gross product in production Activities (A_P), volumes of imported Commodities (A_B), and volume figures of final demand items (A_C, A_G, A_I, A_E). The elements of X are Commodities for final demand not accounted for by the final demand Activities.

5.1 The Activity analysis framework as part of an embracing model

The basic equation (5.1) can be considered as a subset of relations within a greater macro-economic model. (We ignore for a while all dynamic aspects and subsequently dating of variables and coefficients, which otherwise would appear by superscripts, is suppressed). We shall speak in general way of (5.1) or extensions to it as the "inner model" and the remaining part of the whole model as the "outer model". The whole model is said to "embrace" the inner model. This conceptual decomposition need not be meaningful unless the model is specified as a set of equations. This is assumed in the following unless otherwise stated.

In the whole model the variables are either "exogenous" or "endogenous" in the usual sense, while in the inner model the variables are classified as either "given" or "unknown". Given variables are variables of the inner model which are either exogenous or, alternatively, can be determined from the relations of the outer model.

A "simulation" of the simple traditional input-output model which calculates gross production in sectors from a given final demand vector could be attempted by inserting in (5.1) a given Commodity surplus vector and given Activity levels for the final demand Activities. By decomposing and rearranging (5.1) can be written as

(5.2) $\Lambda_B A_B + \Lambda_P A_P = X^* - \Lambda_C A_C^* - \Lambda_G A_G^* - \Lambda_I A_I^* - \Lambda_E A_E^*$

(Exogenous and given vectors are here and in the following indicated by a superscript '*').

In (5.2) there are $(n_{A_B} + n_{A_P})$ unknown variables and n_X relations. Since the number of production Activities (n_{A_P}) is greater than the number of Commodities (5.2) has a positive number of degrees of freedom even if the set of import Activities is empty. The difference from the simple input-output model is that each Commodity may be produced in more than one Activity. Given final demand does not imply a unique solution of (5.2). Additional relations are necessary to determine the distribution of Commodity production between Activities having identical output Commodities (by our definition of Commodity). Such relations may arise from different theoretical assumptions.

Assuming that the whole model is well defined in the sense that all variables can be determined from given values of the exogenous variables, we shall say that the outer model "imposes" on the inner model a number of additional relations to make (5.1) a determinate system. Before going into the contents of such relations we shall look at the formal aspect of this imposition.

On the assumptions that the net surplus Commodity vector is exogenous and the imposed relations are linear the most general formulation of this imposition is a set of linear constraints including constant terms between the Activity levels, equal in number to the degrees of freedom of (5.1), that is $(n_A - n_X)$.

Under these constraints all Activity levels can be written as linear functions of a subset of n_X Activity levels. This is the way we shall formulate the constraints.

(5.3) A_π = "independent Activity levels", Dim $A_\pi = n_X$.

(5.4) ¶ = "linear constraints", Dim ¶ = (n_X, n_A).

The non-zero elements on row i of ¶ indicate how the Activity level i is determined as a weighted sum of independent Activity levels exclusive of a possible given addition to the Activity level (see below).

The subscript of A_π indicates that the choice of independent Activity levels is relative to the choice of the linear constraints matrix. It will be possible, in general, to choose A_π and ¶ in a number of different, but equivalent ways.

(5.5) A_0 = "given additions to Activity levels", Dim $A_0 = n_A$.

The linear constraints can now be written as

(5.6) $A = ¶A_\pi + A_0^*$

By substitution A_π can be eliminated as the following deduction shows. Substitution of (5.6) into (5.1) gives

(5.7) $\Lambda ¶ A_\pi = X^* - \Lambda A_0^*$

Consistency of our assumptions requires $\Lambda¶$ to be non-singular. Necessary conditions for this are that Rank $¶ = n_X$ and Rank $\Lambda = n_X$. The first of these rank conditions expresses that the number of linear constraints does not supercede $(n_A - n_X)$. The second rank condition will certainly be fullfilled for the Ordinary Set of Activities as a consequence of the use made of characteristic Commodity by defining Activity (see section 4.1). The two rank conditions do not, however, imply non-singularity of $\Lambda¶$.

Solving (5.7) with respect to A_π and substitution back into (5.6) gives the following solution of the Activity levels.

(5.8) $A = ¶(\Lambda¶)^{-1}X^* + (I - ¶(\Lambda¶)^{-1}\Lambda)A_0^*$

5.2 Interpretations of linear constraints on the inner model

The linear constraints matrix ¶ of section 5.1 may be given different interpretations. More precisely, different parts of the linear constraints matrix may be interpreted as linear specifications of different relations. Many of the most usual extensions to the traditional input-output model may be incorporated within the linear constraints matrix. Some examples of such extensions are given below.

Linear constraints may be used to determine the relative market shares of Activities with identical output Commodities. Introduction of such constraints in the inner model will dispose of the additional degrees of freedom which the Activity model posesses over the traditional input-output model. The constraints connecting the Activity levels need not be interpreted as anything like "structural constants". The relevant elements of ¶ may be determined by the outer model, for instance as dependent upon the relative costs of identical Commodities produced in different Activities.

There are two important special cases of the interpretation of a submatrix of ¶ as market shares. One of these concerns the treatment of competitive imports. Competitive imports of a Commodity imply coexistence of an import Activity and one or more production Activities with common output Commodity. By means of the linear constraints it will be easy to formulate linear or linearized assumptions about the distribution of Commodity supply between import Activities and production Activities.

The other special case concerns the introduction of technological change in the model.For medium term applications the Activity framework offers a simple way of introducing changes in input-output coefficients. This can be done by adding new production Activities representing "new technology" and letting the linear constraints determine the shares of "old technology" and "new technology". The linear constraints may express trends or indeed any exogenously given development in technological change, or they may express relationships with other variables in the model.

It is also easy to see how the linear constraints in (5.6) can be used to fix Activity levels at given values. The relevant rows of ¶ will have no non-zero elements while A_0 will contain the assigned values. The given values may be exogenous or, alternatively, determined in the outer model.

For the import Activities an application would be to take care of given import quotas. For the production Activities given Activity levels could arise from an assumption of full capacity use of the Activities in question. Alternatively, the Activity levels for production Activities could be determined in the outer model, for instance by means of production functions and capital and labour allocated to the various Activities.

For the final demand Activities there are likewise various possible interpretations of the linear constraints. The final demand Activity levels may, for instance, be fixed at given values. This corresponds to the traditional input-output model with an exogenous final demand vector.

The Activity levels of the final demand Activities may be determined by introducing relations in the model which include both variables of the inner and the outer model. By solving for all variables of the outer model and linearizing, such relations can be "reduced" to linear constraints in the inner model. These linear constraints will be linear functions of the levels of production Activities. For the private consumption Activity the coefficients of such a linear function can be interpreted as composed of a number of factors like real wage income per unit of Activity level, tas rates, and Engel and Cournot derivatives.

5.3 The dual price model

It is well known how the traditional input-output model can be used for computing prices and quantities of gross production in sectors in two separate systems of equations. The quantities will typically be computed for given levels of final demand while prices are typically computed from given prime costs in sectors. The two systems are duals of each other.

A similar approach runs into difficulties with the Activity model. We introduce the following notation for vectors of price indices.

(5.9) p_A = "Activity prices", Dim $P_A = n_A$.

(5.10) p_X = "Commodity prices", Dim $p_X = n_X$.

The Commodity prices are simply price indices of the various Commodities. (We leave the question open whether Commodity prices ought to be buyers' prices, sellers' prices, or any other price concept.) The Activity prices on the other hand are price indices of the Activity levels and the interpretation depends on the type of Activity.

For import Activities the Activity prices are price indices of the imported Commodities. For production Activities the Activity prices are price indices of the gross product in the Activities. We shall refer to Activity prices of production Activities as "prime costs in Activities". For the final demand Activities the Activity prices are price indices of the various final demand items.

The basic equation (5.1) has a dual in price variables.

(5.11) $\Lambda' p_X = p_A$

By decomposing Λ and p_A the dual of (5.2) appears as six separate relations.

(5.12) $\Lambda'_B p_X = p_{A_B}$

(5.13) $\Lambda'_P p_X = p_{A_P}$

(5.14) $\Lambda'_C p_X = p_{A_C}$

(5.15) $\Lambda'_G p_X = p_{A_G}$

(5.16) $\Lambda'_I p_X = p_{A_I}$

(5.17) $\Lambda'_E p_X = p_{A_E}$

Consider (5.13) on the assumption that prime costs in Activities (p_{A_P}) are given. (We leave the question open how to determine the prime costs in Activities.)

(5.18) $\Lambda'_P p_X = p^*_{A_P}$

(5.18) contains n_{A_P} equations in n_X unknown. Since n_{A_P} is greater than n_X the system of equations (5.18) is overdetermined. The economic content of this inconsistency is obvious. Identical Commodities are produced in different Activities. There is, however, no reason to expect prices for the same Commodity calculated by cost addition from independently given prime costs in Activities to be identical.

The price calculation in the Activity model can subsequently not be based on independent price calculations for each production Activity (unless we were inclined to accept that identical Commodities can be offered and sold at different prices).

Alternative price theories will be tried out in the model. The price of a Commodity may be assumed to be fixed by cost calculation in the Activity dominating the supply of the Commodity in question. The price of a Commodity for which there are more than one production Activity with a not negligible share of total supply the price may be assumed to be determined by the least profitable Activity. In this connection it will, of course, be necessary to specify in operational terms what is meant by the "profit of an Activity". For Commodities for which there is an import Activity the world market price may be assumed to determine the domestic market price.

By letting the solutions of the price variables influence the distribution of market shares between Activities the Activity model provides an interrelation between the price and quantity calculation which is absent from the traditional input-output model. This feature can be used to determine not only Commodity production between Sectors but also the distribution between imports and domestic production, as well as the choice of production technique within a Sector with more than one Activity producing the same Commodity.

5.4 The outer model of MODIS IV

With reference to the terminology introduced in section 5.1 the outer model of MODIS IV will consist of a number of submodels or groups of relations. The most important of these are submodels for prices, employment, import, private consumption, and indirect and direct taxation. These submodels will be highly interrelated with each other as well as with the inner model. The main groups of exogenous variables will be, first, most Activity levels of the final demand Activities except the private consumption Activities. Secondly, the exogenous variables will include world market prices, publicly administered prices, main components of wage rates, and rates of direct and indirect taxation.

A basic restriction on MODIS IV is that it has to fit into an administrative set-up which for several years has used earlier MODIS versions in the drafting of "national budgets", one-year Government programs for the development of the total economy. This will imply that MODIS IV has to look very much like MODIS III on the outside.

Compared with MODIS III more weight will be tried to put on the need for medium-term programs (4-7 years). For input-output models this raises the problem of taking into consideration changes in input-output coefficients. In the framework described in this paper changes in input-output coefficients over time can be introduced without relations specifying changes in individual coefficients. The Activity input (Λ^-) and output (Λ^{+}) matrices will be undated matrices and the elements will be estimated for a base year by means of the methods described in section 4. The input and output productivity coefficients (η^{-}, η^{+}) will typically be dated coefficients allowing different ratios between input, output and gross product in production Activities. This will make the Activity matrix (Λ) a dated matrix. Besides, the Activity framework offers a simple way of taking care of technological changes by adding additional Activities representing "new technology" as described in section 5.2.

The input-output model framework presented in this paper will probably be applied not only for MODIS IV, but also for later versions in the MODIS series. It is hoped that there may be available resources for exploration of use of this framework in more "advanced" models. Such models may include production functions in labour and capital, capacity restrictions, and suboptimalisations. Further development will also widening the application area of the model, for instance the treatment of pollution and the regional distribution of production.