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On the interpretation of summary measures for the inverse matrix of the input-output model

(Remarks as discussant on the paper with the above title presented by Messrs. Kjeld Bjerke and P. Nørregaard Rasmussen to the Ninth General Conference of the International Association for Research in Income and Wealth, Lom September 1965)

by

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The mass of information contained in even a moderately sized input-output table is truly formidable. In spite of the ease and speed with which modern computers can pick out the relevant pieces of information for the informed and practically oriented user, methods of abstraction and condensation are still needed. Three important reasons for this need can be mentioned:

1) Not all users have at all times access to a suitably programmed computer.
2) The non-specialist may also draw benefits from the utilisation of information contained in input-output tables, and may only be able to digest it in a condensed form.
3) The academic (non-practical) student of economic structure will need condensed measurements in order to characterise specific structural features of an economy.

Several methods of condensation have been suggested:

First of all the straightforward "inversion" of the coefficient matrix is in itself a form of condensation of the information it contains. However, for the purposes specified above, it will usually not be sufficient.

A crude aggregation of more or less similar sectors in the basic table has been the standard method of condensation. "Triangularisation" and "Block diagonalisation" (Cfr. the Norwegian "National accounts classified by fourteen and five industrial sectors") may provide somewhat more sophistically based methods of aggregation. All the same, aggregation always suppresses information and the question may be raised if alternative condensation methods can be found which suppress less, or less important, information.

One alternative to aggregation, particularly for descriptive purposes is the set of condensed measures of the characteristics of the "inverse coefficient matrix" which was presented by Poul Nørregaard Rasmussen at the International Conference on Input-Output Analysis in Varenna in 1954, and subsequently in his "Studies in Intersectoral Relations" (Amsterdam - København 1956). As far as I can judge it is essentially the same set of measurements which is presented in the present paper.

It is very important that efforts at developing new measures like these should be made, and I think that the suggested solutions are truly ingenious.

Still, these measures do not seem to have achieved any widespread use, and it is my belief that this is not primarily due to the fact that the ideas are not known. I believe the reason is rather that the interpretation is difficult and that the measures do not seem to have any operational meaning, that is to say, that they are of little help to the practical applicant of interindustry analysis.

We may write the system of structural equations

(1) $(I - A)\,x = y$

and its solution

(2) $x = Z\,y,$

using the notation of the paper under discussion, with the exception that we use y instead of the x_D in the paper[1].

Looking at an element of the matrix Z, we have

(3) $Z_{ij} = \frac{dx_i}{dy_j}$, where d signifies a derivation

within the system (1), as distinct from the partial deviation:

1) A = an $m \cdot m$ order matrix of input-output coefficients
x = a vector of total outputs from the m sectors of production
y = a vector of final demand from the same sectors
$Z = (I - A)^{-1}$

(4) $a_{ij} = \frac{\delta x_i}{\delta x_j}$, $1 = \frac{\delta x_i}{\delta y_i}$, $0 = \frac{\delta x_i}{\delta y_j}$ $(i \neq j)$.

Adding over i in (3) gives:

(5) $Z._j = \Sigma_i Z_{ij} = \Sigma_i \frac{d x_i}{d y_j} = \frac{d \Sigma_i x_i}{d y_j}$ Normalising by dividing with $Z.. = \Sigma_i \Sigma_j Z_{ij}$ gives

(6) $U._j = \frac{Z._j}{\Sigma_j Z._j}$

Thus the interpretation of $Z._j$ is increase in total duplicated value of production caused by one unit's increase in final demand from sector j. However, total duplicated value of production is not a very interesting magnitude. In a net input-output table, i.e. in a table with intrasector transactions eliminated, its magnitude depends on the number of sectors. In a gross input-output table its magnitude depends on the institutional subdivisions of production.

Still, the relative measure, $U._j$, has the power of giving a certain ordering of the sectors according to their demands on other sectors. But even this ordering is not entirely satisfactory and measures of "variability" are introduced as supplements. However, if the shortcomings of the primary measures are due to the fact that they measure an uninteresting magnitude, this cannot be much helped by supplementing them with secondary measures, concentrating on the same magnitude.

Turning now to the other group of measures, we have

(7) $Z_{i.} = \Sigma_j Z_{ij} = \Sigma_j \frac{d x_i}{d y_j}$

Here the interpretation becomes particularly uncomfortable. $Z_i.$ "can be considered a measure of the increase in total output for sector no. i needed in order to cope with a unit increase in the deliveries to final demand from each sector". But why indeed should we be interested in the effects of increasing final demand from each sector with the same absolute amount ? The authors realize this and considers instead the effects of a proportionate increase in all final demands, but this leads to the trivial effect of proportionate increases in all magnitudes:

Prescribing:

(8) $\frac{\delta y_j}{\delta y.} = \frac{y_j^o}{y_.^o}$ (The o's indicate "base year" values)

we have

(9) $\frac{d x_i}{d y.} = \Sigma_j \frac{d x_i}{d y_j} \cdot \frac{\delta y_j}{\delta y.} = \Sigma_j \frac{d x_i}{d y_j} \cdot \frac{y_j^o}{y_.^o} = \frac{1}{y_.^o} \Sigma_j \frac{d x_i}{d y_j} \cdot y_j^o = \frac{x_i^o}{y_.^o}$

(We also have $\frac{d x_i}{d y.} = \frac{x_i^o}{y_.^o} = \frac{x_i^o}{y_i^o} \cdot \frac{y_i^o}{y_.^o}$ so that the fraction $\frac{x_i^o}{y_.^o}$ or its components $\frac{x_i^o}{y_i^o}$ and $\frac{y_i^o}{y_.^o}$, all directly observable from the original flow matrix have a particular significance as measures of the inverted matrix)

As alternatives to the weighting by final demand we are now given certain components of final demand as weights, e.g. consumer goods deliveries (C) investment goods deliveries (I) or export goods deliveries (A). Writing C_j for consumer goods deliveries from sector no. j and assuming

(10) $$\frac{\delta y_j}{\delta C_\cdot} = \frac{C_j^o}{\Sigma_j C_j^o}$$ we obtain

(11) $$Z_{i.}^C = \Sigma_j \frac{d\, x_i}{d\, C_\cdot} = \Sigma_j \frac{d\, x_i}{d\, y_j} \cdot \frac{\delta y_j}{\delta C_\cdot} = \frac{\Sigma_j \frac{d\, x_i}{d\, y_j} \cdot C_j^o}{\Sigma_j C_j^o}$$

the increase in gross production in sector i per unit proportionate increase in final deliveries of consumer goods. This measure has a specific meaning and a clear significance. Corresponding measures may be calculated for other components, like investment goods and export goods. But when these measures are "weighted" by division with averages, giving:

(12) $$U_{i.}^C = \frac{Z_{i.}^C}{\frac{1}{m} \Sigma_i Z_{i.}^C}$$

I can now longer follow. The reason is again that sums over all sectors of duplicated value of production or its average per sector is uninteresting i.e. whereas $Z_{i.}^C$ is interesting $\frac{1}{m} \Sigma_i Z_{i.}^C$ is uninteresting.

Starting from measures like $Z_{i.}^C$, $Z_{i.}^I$ and $Z_{i.}^A$ I believe we can derive a compact and meaningful set of measures of the inverse of an input-output matrix. We need of course not restrict ourselves to the three extensive complexes specified above, but may consider a somewhat finer breakdown e.g. private consumption of food, of clothing etc. public non defence consumption a.s.o. Suppose we substitute the original m-dimensional vector of final demand (y_i) $(i = 1,2, \ldots, m)$ with an M-dimensional vector (Y_I) $(I = 1,2, \ldots, M)$ $M < m$ defining y_{ik} such that

(13) $$\sum_{K=1}^{M} y_{iK} = y_i$$ and

(14) $$y_{iK} = b_{iK} Y_K,$$ assuming

(15) $$b_{iK} = \frac{y_{iK}^o}{Y_K^o}$$ (the o's indicating "base year" values)

Writing the m x M matrix $(b_{iK}) = B$, we now have

(16) $$(I - A)\, x - B\, Y = 0$$

(17) $$x = [(I - A)^{-1} B]\; Y$$

Here the "inverse" matrix $[(I - A)^{-1} B] = (Z_i^K)$ has only $m \cdot M$ elements instead of the m^2 elements of $(I - A)^{-1}$. This implies savings in comprehension as well as in computations[1)] (if $(I - A)^{-1}$ is not needed for other reasons).

1) Computation of $(I - A)^{-1}$ requires in principle a number of multiplications of the order m^3, whereas computation of $(I - A)^{-1} B$ requires a number of the order $m^2 \cdot M$.

I have maintained that total duplicated value of production, and also sums over all sectors of changes in duplicated value are uninteresting. Sums over specific sectors may nevertheless be of interest: Questions like: How much will gross production in all sectors in food manufacturing have to increase if private consumption of food is increased by one unit proportionately distributed over all items ? may be of interest. Let us assume therefore that we can construct an (N m) aggregation matrix, E, with $N < m$, by which the production figures for the m sectors of the input-output table are aggregated to sums for each of N aggregate sectors, i.e.

(18) $X = E\,x$

We then have

(19) $X = \left\{E\,\left[(I - A)^{-1}\,B\right]\right\}\,Y$

Here $\frac{dX}{dY} = \left\{E\,\left[(I - A)^{-1}\,B\right]\right\}$ is of dimension $N \cdot M$. By choosing convenient dimensions N and M we may in this way obtain a very useful condensed description of the inverse matrix.

In the above process the elements of y are determined by the elements of Y through their assumed fixed proportions. This is the only assumption which is introduced in order to perform the condensation. It may be compared to the usual procedure for aggregating an $(m \cdot m)$ matrix to an $(N \cdot N)$ matrix.

Even in this case we obtain aggregated production values by adding:

(20) $X = E\,x$

Then we assume fixed proportions between the production levels belonging to the same aggregate, getting

(21) $x = D\,X = D\,E\,x$

where D is the matrix of fixed proportions

From

$(I - A)\,x = y$

we now obtain

(22) $E\,x - E\,A\,D\,E\,x = E\,y$

(23) $(I - E\,A\,D)\,(E\,x) = E\,y$

We may now write

(24) $Y = E\,y$

and get the solution in aggregate values

(25) $X = (E\,x) = (I - E\,A\,D)^{-1}\,Y$

We must then be aware that our assumptions about constant aggregate coefficients also implies specific values for the elements of y, namely:

(26) $y = (I - A)\,x = (I - A)\,D\,X = (I \;\; A)\,D\,(I \;\; E\,A\,D)^{-1}\,Y$

This may imply rather peculiar relationships between the elements of Y and the corresponding elements of y. It should be compared with the simple connection:

$$y_{iK} = \frac{y^{o}_{iK}}{Y_K^{\,o}} \cdot Y_K \quad ,$$

implied by (14) and (15) above.

Or, expressed in another way: Since the use of an aggregated coefficient matrix implies an assumption that all the production levels of detailed sectors within the same aggregate sector change in the same proportion, and since intersector deliveries are determined by the economic structure, the final deliveries will have to be adjusted in such a way that this proportionality is attained.

The condensation procedure advocated above therefore appears to give a much better controlled approach than traditional aggregation.

Of even greater interest than the effects on gross production in sectors or groups of sectors may be the effects on the various elements of value added.

Suppose that we have for the elements of value added

(27) $W = F\,x$

where

W = vector of value added elements

F = matrix of "value added" coefficients, e.g. "wages" per unit value produced in industry no. i.

From (16) we now get

(28) $W = F\,[(I - A)^{-1}\,B]\;\;Y$

If W has r elements, then $[F\,(I - A)^{-1}\,B]$ has r M elements.

It may be pointed out that the matrix $[F\,(I \;\; A)^{-1}\,B]$ may be found to be a very useful tool in attempts to use the results of input-output analysis in combination with more aggregate models of the structure in other parts of the economy. Such models must be concerned with the elements of final demand Y, and of value added W. The condensed matrix $[F\,(I \;\; A)^{-1}\,B]$ gives the relationships through the production system, and can be used directly in conjunction with other coefficients representing the relationships through income and demand reactions in the economy.