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Abstract:
In this paper we will derive the optimal solution to a wide class of stochastic optimal environmental preservation problems, taking the quasi-option value into account. The paper generalizes and extends previous results in this area. The optimal policy is to preserve until marginal environmental benefits reaches some trigger level. A striking feature of the optimal policy, is that it is independent of what is assumed about marginal environmental benefits below this trigger level.

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1 Introduction

The quasi-option value of environmental preservation has been recognized since the seminal works of Arrow and Fisher (1974) and Henry (1974), who studied two-period models. Recently, the study of irreversible decisions under uncertainty has made considerable progress through the theory of real options, using methods from stochastic analysis, see Dixit and Pindyck (1994). The methods developed in this literature makes it possible to construct important extensions of the quasi-option models.

The real option theory has been applied to environmental preservation in several studies. Clark and Reed (1990) and Reed and Ye (1994) consider the choice between land development and wilderness preservation, Reed (1993) and Conrad and Ludvig (1994) consider the harvest of old-growth forest, while Conrad (1992) study the accumulation of stock pollutants. In the current paper we will present a model that generalizes many of these models from the literature, and provide a general solution to this class of models. The optimal policy is to preserve the natural capital until the marginal environmental benefits reaches some lower trigger level. A striking feature of the optimal policy is that it is independent of what is assumed about marginal benefits below the trigger level.

The paper is based on previous work by Kobila (1991,1993) who considered the optimal increases in capital stock under uncertainty. In their model the capital stock could only increase. Arntzen (1995) extended Kobila's model to allow reduction in capital stock, without recovering all investment cost. Adding some extra assumptions he was able to derive an extension of Kobila's result. In this paper we will consider the case where the capital stock can only decrease.

2 The model

Let K_t denote the stock of some environmental resource. This may be the stock of remaining rain forest, the area of some type of wilderness or the atmospheres remaining capacity of absorbing greenhouse gasses. Let Θ_t denote the general attitude toward this environmental resource. This attitude determines the valuation of the benefits $\pi(\Theta_t, K_t)$ from this resource. It is supposed that the marginal benefit, with respect to the stock of the resource, is increasing in[1] θ and decreasing in k, i.e. $\frac{\partial^2 \pi}{\partial \theta \partial k} \geq 0$ and $\frac{\partial^2 \pi}{\partial k^2} \leq 0$. We also assume, given a $K_0 < \infty$, that π is either bounded,

$$\sup_{\theta \in R_+, k \leq K_0} \pi(\theta, k) < \infty$$

or linear in θ (at least for θ sufficiently large)

$$\pi(\theta, k) = \theta \lambda(k) - \xi(k)$$

In the case of K_t as remaining rain forest, $\lambda(k)$ will be the value of the environmental amenity, whereas $\xi(k)$ is the opportunity cost, the value the same area would have if developed e.g. for agriculture.

We model Θ_t as a geometric Brownian motion

$$d\Theta_t = \alpha \Theta_t dt + \beta \Theta_t dB_t$$

where B_t is a Brownian motion. The extraction $u_t \geq 0$ of the resource is the negative of the change in the stock, thus $dK_t = -u_t dt, K_t \geq 0$. This extraction will give some short term benefits $P(K_t)u_t$, where $P(K_t)$ is interpreted as the price of the extracted resource, e.g. the timber price, if the natural capital is a forest. $P(K_t)$ is allowed to depend on the stock of capital, and we assume that the most valuable resource is extracted first, thus $P'(k) \geq 0$. The optimal total

1

We use capital letters (K_t, Θ_t etc.) to denote stochastic processes, while lower case letters (k, θ etc) are used to denote the possible values of these processes.

benefits are thus

$$H(t,\theta,k) = \sup_{u_s \geq 0, K_s \geq 0} E^{t,\theta,k} \int_t^{\infty} (\pi(\Theta_t, K_t) + P(K_t)u_t)e^{-rt}dt, \tag{1}$$

where $r > \alpha$ is the discount rate.

We consider the case where resource extraction is irreversible, like harvest of old-growth forest, development of waterfalls to hydro-electric power plants or development in wilderness areas, etc. Thus K_t is non-increasing, or equivalently $u_t \geq 0$.

A similar problem was considered by Kobila (1991,1993) where K_t were assumed to be non-decreasing. Formally, a problem with non-decreasing capital stock can readily be transformed into one with non-increasing capital stock, since problems with non-increasing capital can be reformulated into one with non-decreasing capital, simply by introducing $X_t = 1 - K_t$ as a new state variable. Thus it may appear that problems with decreasing capital are included in the previous solution. This in not true, since the properties of the benefits and cost functions are changed in the process of reformulating the problem. As above, if $\pi(\theta, k)$ is increasing in k, the benefits will be decreasing in $x = 1 - k$. Kobila's solution of the problem of optimizing (1) with K_t non-decreasing, assumed that π is increasing in k, thus Kobila's theorem does not cover this case. Still, their approach generalizes to this problem as well. This generalization is the purpose of the paper. A further purpose of the paper is to study how the solution depend on the form of π.

Kobila's result was later extended by Arntzen (1995) who considered a case where K_t could both increase and decrease, but where decreasing K_t were interpreted as selling capital equipment, and the selling price was assumed to be less than the purchasing price. Letting the purchasing price go to infinity, this model would provide a solution to the problem of this paper, but at the cost of adding many assumptions not needed here.

3 The optimal policy

The discussion in this section is heuristic. To make the discussion accessible also for readers not familiar with stochastic analysis, all proofs are delegated to the appendix. The technically inclined reader may want to confront Kobila (1993) for a discussion to some of the problems left out in this paper.

The solution is stated in the next theorem, where $\gamma_1 > 0 > \gamma_2$, are the roots of the characteristic equation

$$-r + (\alpha - \frac{1}{2}\beta^2)\gamma + \frac{1}{2}(\beta\gamma)^2 = 0.$$

Theorem 1 *Let Θ_t be a geometric Brownian motion as given above. Moreover, assume that there exists $M < \infty$ such that*

$$\tilde{\pi}(\theta, k) - \pi(\theta, k) \leq M$$

where $\tilde{\pi}$ is defined by

$$\tilde{\pi}(\theta, k) = \begin{cases} \pi(\theta, k) \text{ if } k \leq \phi(\theta) \\ \pi(\theta, \phi(\theta)) + r\int_{\phi(\theta)}^{k} P(x)\, dx \text{ if } k > \phi(\theta) \end{cases}, \tag{2}$$

then the solution to

$$H(t, \theta, k) = \sup_{u_s \geq 0, K_s \geq 0} E^{t,\theta,k} \int_t^\infty (\pi(\Theta_s, K_s) + P(K_s)u_s)e^{-rs}ds$$

is given by $H = h$, where

$$h(t, \theta, k) = \frac{2e^{-rt}}{(\gamma_1 - \gamma_2)\beta^2} \left[-\theta^{\gamma_1} \int_\infty^\theta \frac{\tilde{\pi}(s, k)}{s^{\gamma_1+1}} ds + \theta^{\gamma_2} \int_0^\theta \frac{\tilde{\pi}(s, k)}{s^{\gamma_2+1}} ds \right].$$

The corresponding optimal control is

$$u^*(t, \theta, k) = \begin{cases} \infty \text{ if } \theta \leq \psi(k) \\ 0 \text{ otherwise} \end{cases}$$

where $\psi(k)$ is determined by the equation

$$rP(k) = \gamma_1 \psi(k)^{\gamma_1} \int_{\psi(k)}^\infty \frac{\pi_k'(s, k)}{s^{\gamma_1+1}} ds. \tag{3}$$

These results also hold if

$$\pi(\theta, k) = \theta\lambda(k) - \xi(k) \tag{4}$$

with $\lambda^{'}(k) \geq 0, \lambda^{''}(k) \leq 0$ *and* $\xi'(k) \geq 0$. *In this case the (3) simplifies to*

$$rP(k) = \frac{\gamma_1}{\gamma_1 - 1}\psi(k)\lambda'(k) - \xi'(k)$$

This solution implies that for each level of the natural capital stock there is a reservation level $\psi(k)$, such that we do not extract the resource until Θ_t fall below this level. We extract the resource once we reach this level, and the stock of capital is reduced to a level below k. With the given assumptions, the curve $\psi(k)$ will be increasing, hence when k is reduced, the reservation level $\psi(k)$ reduced too, and we should wait for a further reduction in Θ_t to extract more of the resource. Graphically, the solution can be illustrated by defining a 'extraction area'

$$\mathcal{A} = \{(\theta, k) : 0 \leq \theta < \psi(k); k \geq 0\}$$

whenever the process (Θ_t, K_t) reaches the boundary of this area, K_t is reduced to keep the process outside the area, while the process Θ_t is unaffected.

The solution may be illustrated as in Figure 1. The area below the $\psi(k)$ curve is called the preservation area, where no development or extraction is optimal, i.e. $u^* = 0$. Above the $\psi(k)$ curve extraction is optimal, and we call it the extraction area. If the process start in this area, the optimal policy will be to reduce K_t immediately down to the curve. When the process starts in the preservation area, the stock will be held constant, and the process (Θ_t, K_t) will 'live' on a horizontal line until it reaches the $\psi(k)$ curve. At the curve the process is reflected downward, and once it drops off the curve it starts living on a new horizontal line, etc.

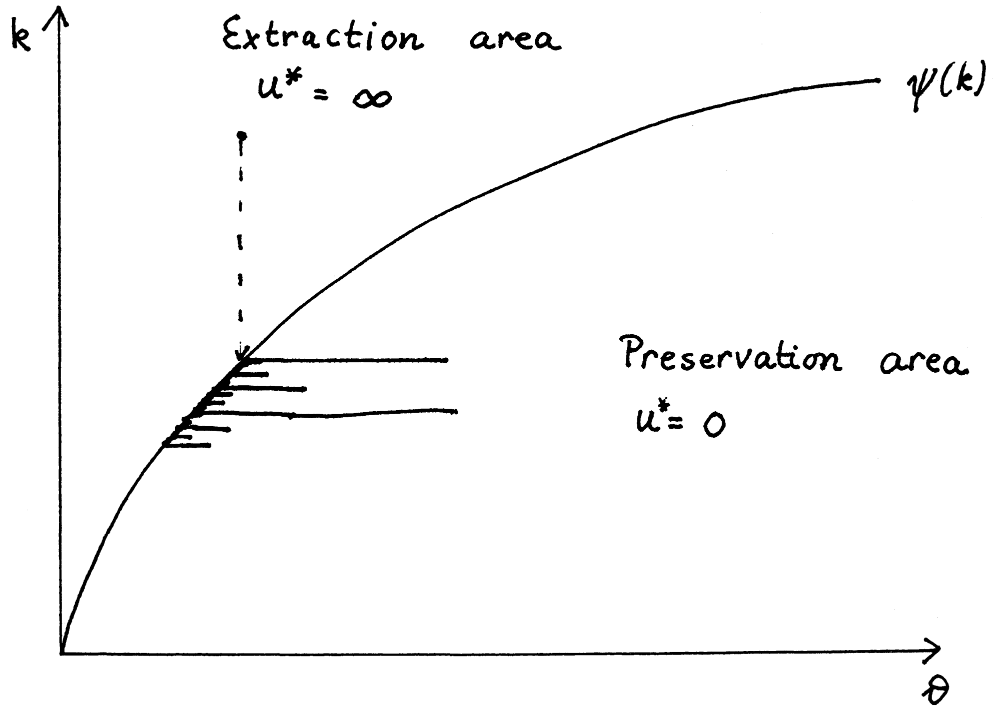

Figure 1: Illustration of the optimal policy

For a complete solution of the problem, we need to determine the reservation level $\psi(k)$. This level is determined by the equation (3). Solving this equation for ψ, may appear very complicated, but note that solving (3) for $\psi(k)$ is much simpler than solving the original stochastic optimization problem. Actually, in many cases it is possible to compute the integral analytically, which simplifies the equation considerably. In the particular case of π linear, we see that a rather explicit solution is possible. In more general cases, (3) may easily be numerically solved k-vise for $\psi(k)$.

Note that (3) can be rewritten as

$$P(k) = Q(\psi(k), k).$$

where

$$Q(\psi, k) = \frac{\gamma_1}{r}\psi^{\gamma_1}\int_{\psi}^{\infty}\frac{\pi_k'(s,k)}{s^{\gamma_1+1}}ds,$$

may be interpreted as a reservation price. Given k and $\theta = \psi$, it is optimal to extract the resource only if the resource price $P(k)$ exceeds the reservation price. As stated below, and proven in the appendix, Q is non-decreasing in ψ, and strictly increasing unless $\pi'_k(\theta, k)$ is constant for $\theta \geq \psi$.

3.1 Irrelevant marginal environmental benefits in $\mathcal{A}$

A striking feature of the optimal policy, is that the equation determining $\psi(k)$ is independent of $\pi(\theta, k)$ for $\theta < \psi(k)$. The only restriction on π in this area is that $\pi''_{\theta k} \geq 0$, and hence that the marginal environmental benefits $\pi'_k(\theta, k)$ for $\theta < \psi(k)$, are restricted above by $\pi'_k(\psi(k), k)$. Whether the benefits from the environment cannot fall lower, ($\pi''_{\theta k}(\theta, k) = 0$ for $\theta < \psi(k)$), or the benefits are likely to fall rapidly if Θ_t is declining, ($\pi''_{\theta k}(\theta, k)$ is very large for $\theta < \psi(k)$) has no effect on the optimal preservation of the next marginal unit of environment. The reason for this independence is that the process never enters the extraction area, and if it happens to start there, we will immediately leave the area. If we decide to preserve a particular acre of forest only as long as $\Theta_t \geq \psi$, we will never get any environmental benefits from this forest when $\Theta_t < \psi$.

Still, the independence is striking, as the future environmental benefits forgone by extracting the resource when $\Theta_t = \psi$ depends on π'_k also for $\Theta_t < \psi$. On the other hand, the alternatives are not to extract the resource now or newer, but now or later – perhaps never. If we restrict the attention to the class of trigger price strategies, the alternatives are to extract it at $\Theta_t = \psi$, or at some other level $\hat{\psi}$, i.e. when $\Theta_t = \hat{\psi}$. Since π'_k is a continuous function of θ (we even assume that $\pi''_{k\theta}$ exist), the effect of a marginal shift in ψ, will be determined by $\pi'_k(\psi, k)$. Given the assumptions on P and π, the first order conditions actually gives the optimal solution.

3.2 Uncertainty and irreversibility

To study the effect of irreversibility and uncertainty and to further explain why only π'_k outside the area $\mathcal{A}$ matters, consider the case of full certainty. We have to treat $\alpha > 0$ and $\alpha \leq 0$ separately. First the case that $r > \alpha > 0$. In this case the environmental benefits are nondecreasing over time. If it is not optimal to extract the resource now, it certainly will not be optimal in the future. The extraction is thus optimal if and only if the resource price exceed present value of environmental benefits.

We first show that this rule is consistent with the optimal policy given in theorem 1. At the state $\theta = \psi(k)$, the present value of future benefits from the marginal capital at stock k is

$$\int_0^\infty \pi_k'(\psi(k)e^{\alpha t}, k)e^{-rt}dt.$$

Changing the variable of integration to $s = \psi(k)e^{\alpha t}$, we find that extraction is optimal if the resource price exceed the reservation price

$$\int_0^\infty \pi_k'(\psi(k)e^{\alpha t}, k)e^{-rt}dt = \frac{1}{\alpha}\psi^{\frac{r}{\alpha}} \int_{\psi(k)}^\infty \frac{\pi_k'(s,k)}{s^{\frac{r}{\alpha}+1}}ds.$$

Letting $\beta \longrightarrow 0$ we find that $\gamma_1 \longrightarrow \frac{r}{\alpha}$, and thus the reservation price is identical to the one in the theorem.

As stated in the next proposition, with uncertainty, the requirement that the present value should be positive, is strengthened. We will not extract the resource unless the price is as least as high as the reservation price, and where the reservation price is strictly higher than the forgone environmental benefits. Note that this result also applies for all $\alpha < r$.

Proposition 2 *Q is non-decreasing in ψ, and strictly increasing unless $\pi_k'(\theta, k)$ is constant for $\theta \geq \psi$. Moreover, if the process starts at $\Theta_0 = \theta$, for $\theta \leq \psi(k)$, the present value of future marginal environmental benefits must be strictly less than the reservation price. That is, for $\theta \leq \psi(k)$,*

$$Q(\theta, k) > E^{\Theta_0 = \theta}\{\int_0^\infty \pi_k'(\Theta_t, k)e^{-rt}dt\}.$$

How much should the reservation price exceed the present value of forgone benefits? Remember that the reservation price only depends on the marginal benefits for $\theta \geq \psi$. On the other hand, the expected future benefits from a unit of environmental capital obviously depends on $\pi'_k(\theta, k)$ for all θ. It follows that the reservation price cannot exceed the expected benefits by some constant mark-up factor.

To further characterize the reservation price, we rewrite it as

$$Q(k, \psi) = \frac{\gamma_1}{r} \int_{\psi(}^{\infty} \left(\frac{\psi}{s}\right)^{\gamma_1} \frac{\pi'_k(s, k)}{s} ds.$$

It is easy to verify that

$$\int_{\psi}^{\infty} \left(\frac{\psi}{s}\right)^{\gamma_1} \left(\frac{\gamma_1 - 1}{\psi}\right) ds = 1,$$

and hence $\left(\frac{\psi}{s}\right)^{\gamma_1} \left(\frac{\gamma_1 - 1}{\psi}\right)$ defines a probability density over the interval $[\psi, \infty)$. Let $\hat{E}$ denote the expectation operator with respect to this density. Then we get

$$rQ(k, \psi) = \frac{\gamma_1}{\gamma_1 - 1} \psi \hat{E} \left[\frac{\pi'_k(s, k)}{s}\right].$$

Changes in the uncertainty will both affect the probability density, and hence the expectation, but also the coefficient $\frac{\gamma_1}{\gamma_1 - 1}$. The effect of increasing β on $\hat{E}\left[\frac{\pi'_k(s,k)}{s}\right]$ depends on the function π. In the linear case $\pi(\theta, k) = \theta\lambda(k)$, this expectation is independent of β. In general it may be both increasing and decreasing. The coefficient $\frac{\gamma_1}{\gamma_1 - 1}$ is well-known from the real option literature, and is increasing in β. The dependence between β and $\frac{\gamma_1}{\gamma_1 - 1}$ is illustrated in Figure 2.

Let us return to the case of full certainty, and consider the case $\alpha \leq 0$. In this case the value of the environment will be non-increasing over time. If extraction is not optimal now, it may become optimal in the future. The question is whether it is optimal to delay investment for a period, Δt. The delay will give environmental benefits $\pi'_k(\theta, k)\Delta t$ at the cost of the return to the value of the resource, $rP(k)\Delta t$. Thus unless, $rP(k) > \pi'_k(\theta, k)$, delay is optimal, and we will invest once $\theta = \psi$, where $rP(k) = \pi'_k(\psi, k)$. (This is consistent with the theorem, as $\hat{E}(\pi'_k(s, k)/s) = \pi'_k(\psi, k)/\psi$, when $\beta \to 0$ and $\alpha \leq 0$.) Note also that this rule would apply even under uncertainty, if the extraction of the resource is

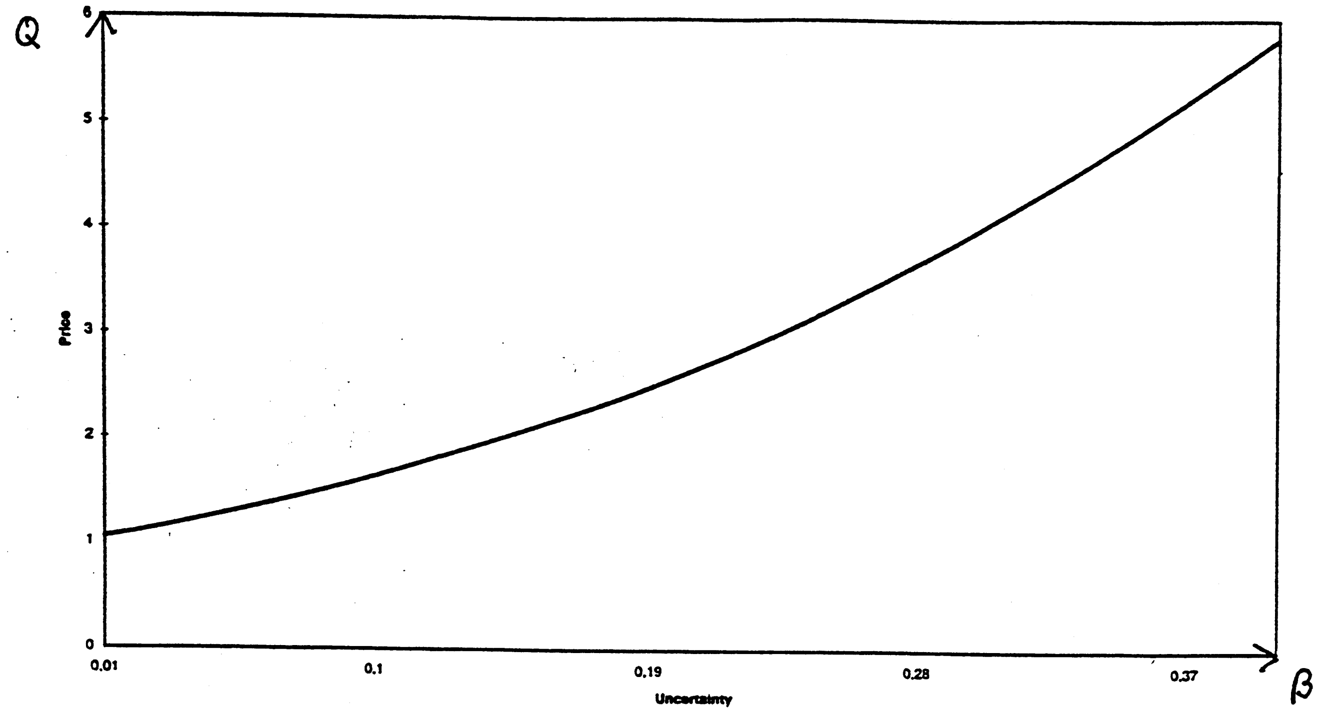

Figure 2: The uncertainty coefficient as a function of β

fully reversible. With reversible extraction, we can choose between environmental benefits, π'_k or return to the resource value rP during the period Δt, irrespective of what happens at other points in time. Thus with certainty and $\alpha \leq 0$ or if extraction is reversible, we extract for $\Theta_t \leq \psi$ where $rP(k) = \pi'_k(\psi, k)$. The next proposition shows that this rule to is strengthened in the case of uncertainty and irreversibility, and again the result applies for all $\alpha < r$.

Proposition 3 *Suppose that π'_k is not independent of θ, and $\beta > 0$. Then, preservation is optimal whenever the marginal benefit stream from the environmental resource is higher than the interest payment on the resource price would be, or equivalently*

$$rP(k) > \pi'_k(\theta, k) \text{ for all } \theta \leq \psi(k)$$

Note that the reverse is in general not true: even if $rP(k) \leq \pi'_k(\theta, k)$, preservation may still be optimal. The proposition above thus shows that we will be more prone to preserve the environment, when the destruction of it will be irreversible, than when it is reversible.

As the uncertainty increases, we would expect the option value to increase, that is increasing uncertainty should increase the cost of taking an irreversible action. In the present model, this would imply that the boundary of the preservation set, $\psi(k)$, should move to the left as the uncertainty increase. This turns out to be true:

Proposition 4 *$\psi(k)$ decreases as β increases.*

3.3 Lower bond on natural capital

The stochastic process Θ_t may eventually take very small values. As Θ_t is a geometrical Brownian motion, it is known that if $2\alpha < \beta^2$, then, with probability 1, $\Theta_t \to 0$ as $t \to \infty$. What happens to the natural capital under the optimal strategy, as $\Theta_t \to 0$? Under what conditions will it be optimal to keep a fraction of the natural capital even when the marginal benefits are at the lowest?

Note first that the reservation price $Q(\psi, k)$ is increasing in ψ. Hence, if development is not optimal for low values of Θ_t it never will be optimal. Given a capital stock k, further reduction in the stock will not be optimal if $P(k) \leq \lim_{\psi \to 0} Q(\psi, k)$. Since $\lim_{\psi \to 0} Q(\psi, k) = \lim_{\psi \to 0} \frac{1}{r}\pi'_k(\psi, k) = \frac{1}{r}\pi'_k(0, k)$, further development will not be optimal when $rP(k) \leq \pi'_k(0, k)$. Moreover, $\pi''_{kk} \leq P'(k)$, and hence the next proposition follows

Proposition 5 *There is an lower bound $\bar{k}$ on the optimal natural capital stock, such that further development will not be optimal if $k \in [0, \bar{k}]$ even if $\Theta_t \to 0$. The lower bound is determined as*

$$\bar{k} = \max\{k : rP(k) \leq \pi'_k(0, k)\}$$

Note that the lower bound may be zero. If $P(k) > \pi'_k(0, k)$ for all $k > 0$, then the optimal stock of natural capital will approach zero as $\Theta_t \to 0$.

We note that this lower bound on the natural capital is independent of the parameters in the stochastic process, and hence not influenced by the degree of uncertainty. The intuitive explanation of this result is that since Θ_t is a geometrical Brownian motion $d\Theta_t$ is proportional to Θ_t, and hence if Θ_t reaches a low value, it will change very slowly, and thus essentially become stuck if it reaches low values. As $\Theta_t \to 0$, the problem approaches a deterministic problem with constant future marginal environmental benefits equal $\pi'_k(0, k)$. Preservation is optimal if the present value of these benefits, $\pi'_k(0, k)/r$, exceeds the value of the resource $P(k)$.

4 Conclusions

In this paper we have solved a quite general model of optimal environmental preservation under uncertainty. The generality allows us to study the effect of different assumption about marginal environmental benefits π'_k. A striking feature of the optimal solution is the independence of the marginal benefits for values of the stochastic parameter below the reservation value, i.e. for $\theta \leq \psi(k)$.

In the case of full certainty, we distinguished between the cases $\alpha > 0$, and $\alpha \leq 0$. In both cases only $\pi'_k(\theta, k)$ for $\theta \geq \psi$, matters, but the preservation criterion were different. When $\alpha > 0$, we invest when the price is higher than or equal to the present value of future benefits, while with $\alpha \leq 0$, we invest once the environmental benefits are less than the return to the value of the resource. With uncertainty both these rules applies simultaneously, but they are strengthened. The reservation price is higher than the present value of marginal environmental benefits and the next proposition shows that the reservation price also gives a return strictly higher than the marginal benefits of the environment.

A Proofs and Lemmas

The proof of theorem 1 requires the following lemma.

Lemma 6 *Suppose there exists a continuous function $k = \phi(\theta) \geq 0$ and a function $h(t, \theta, k)$ which is C^1 in t and k, and C^2 in θ, and such that*

$$\sup_{u \geq 0}[L^u h + (\pi + Pu)e^{-rt}] \begin{cases} = 0 \text{ if } k \leq \phi(\theta) \\ \leq 0 \text{ if } k > \phi(\theta) \end{cases} \tag{A.5}$$

and

$$\frac{\partial h}{\partial k} - Pe^{-rt} \begin{cases} > 0 \text{ if } k < \phi(\theta) \\ = 0 \text{ if } k \geq \phi(\theta) \end{cases}$$

Moreover, suppose that there exists $M < \infty$ such that for all $v \geq 0$, $t \geq 0$ and $k > \phi(\theta)$

$$L^v h(t, \theta, k) + (\pi(\theta, k) + P(k)v)e^{-rt} \geq -M$$

and that for all t,θ,k and all controls $u \geq 0$

$$\lim_{T \to \infty} E^{t,\theta,k}[h(Y_T^u)] = 0$$

where Y_t is the state of the system, defined by $Y_t = (t, \Theta_t, K_t)$. Then $h = H$, and the optimal policy is

$$u^*(t, \theta, k) = \begin{cases} \infty \text{ if } k \geq \phi(\theta) \\ 0 \text{ if } k < \phi(\theta) \end{cases}$$

Proof. The proof of this theorem follows Kobila (1991) closely. The proof in Kobila (1991) requires an upper bound on the capital extension, in our model the corresponding condition is automatically satisfied as we have assumed that $K_t \geq 0$. □

Proof. of theorem 1. (a sketch) As in Kobila (1991) we write $G(\theta, k) := H(t, \theta, k)e^{rt}$, and find, using the H-J-B equation (5), that G has to satisfy

$$-rG + \alpha\theta\frac{\partial G}{\partial \theta} + \frac{1}{2}(\beta\theta)^2\frac{\partial^2 G}{\partial \theta^2} = -\tilde{\pi}(\theta, k)$$

with boundary conditions $G(0, k) = \tilde{\pi}(0, k)/r$ and $G(\infty, k) = \tilde{\pi}(\infty, k)/r$ where

$$\tilde{\pi}(\theta, k) = \begin{cases} \pi(\theta, k) \text{ if } k \leq \phi(\theta) \\ \pi(\theta, \phi(\theta)) + r \int_{\phi(\theta)}^{k} P(x)\, dx \text{ if } k > \phi(\theta) \end{cases} \tag{A.6}$$

The solution to this differential equation can be derived by Lemma 1 in Kobila (1991), as

$$G(\theta, k) = \frac{2}{(\gamma_1 - \gamma_2)\beta^2} \left[-\theta^{\gamma_1} \int_{\infty}^{\theta} \frac{\tilde{\pi}(s,k)}{s^{\gamma_1+1}} ds + \theta^{\gamma_2} \int_{0}^{\theta} \frac{\tilde{\pi}(s,k)}{s^{\gamma_2+1}} ds \right],$$

where $\gamma_1 > 0 > \gamma_2$, are the roots of the characteristic equation given by

$$-r + (\alpha - \frac{1}{2}\beta^2)\gamma + \frac{1}{2}(\beta\gamma)^2 = 0$$

At this point is more convenient to specify the boundary of $\mathcal{A}$ using the inverse of the function ϕ, thus let $\psi(k) = \phi^{-1}(k)$, and note that at this boundary

$$G'_k(\psi(k), k) - P(k) = 0.$$

Inserting the explicit solution above, and using the definition of $\tilde{\pi}$, we find

$$\begin{aligned} P(k) &= G'_k(\psi(k), k) \\ &= \tfrac{2}{(\gamma_1-\gamma_2)\beta^2} \left[-\psi(k)^{\gamma_1} \int_{\infty}^{\psi(k)} \tfrac{\pi'_k(s,k)}{s^{\gamma_1+1}} ds + \psi(k)^{\gamma_2} \int_{0}^{\psi(k)} \tfrac{rP(k)}{s^{\gamma_2+1}} ds \right] \\ &= \tfrac{2}{(\gamma_1-\gamma_2)\beta^2} \left[-\psi(k)^{\gamma_1} \int_{\infty}^{\psi(k)} \tfrac{\pi'_k(s,k)}{s^{\gamma_1+1}} ds - \tfrac{rP(k)}{\gamma_2} \right] \end{aligned}$$

Collecting the $P(k)$ terms we find

$$P(k) \left[1 + \frac{2r}{\gamma_2(\gamma_1 - \gamma_2)\beta^2} \right] = \frac{2}{(\gamma_1 - \gamma_2)\beta^2} \psi(k)^{\gamma_1} \int_{\psi(k)}^{\infty} \frac{\pi'_k(s,k)}{s^{\gamma_1+1}} ds$$

Rearranging we get

$$P(k) = \frac{2\gamma_2}{\gamma_2(\gamma_1 - \gamma_2)\beta^2 + 2r} \psi(k)^{\gamma_1} \int_{\psi(k)}^{\infty} \frac{\pi'_k(s,k)}{s^{\gamma_1+1}} ds.$$

Using the equation $\gamma_2\gamma_1\beta^2 = -2r$, this simplifies to

$$rP(k) = \gamma_1 \psi(k)^{\gamma_1} \int_{\psi(k)}^{\infty} \frac{\pi'_k(s,k)}{s^{\gamma_1+1}} ds.$$

This proves the theorem. □

Proof. of Proposition 3: A formal proof of the claim can be derived from applying Dynkin's formula. Suppose that $rP(k) < \pi'_k(\theta, k)$ at some point $(\theta, k) \in \mathcal{A}$. We claim that we can find an open neighborhood $O \subset \mathcal{A}$ such that $rP(k) < \pi'_k(\theta, k)$ for all $(\theta, k) \in O$ and the line-segment from (θ, k) to $(\theta, \phi(\theta))$ is in the closure of O. To see this, consider first the case that $\psi(k) < \infty$. Since $\frac{\partial^2 \pi}{\partial\theta\partial k} \geq 0$, the inequality $rP(k) < \pi'_k(\theta, k)$ continues to hold as θ increases. Thus we may extend the neighbourhood O to include $\psi(k)$. Next suppose that $\psi(k) = \infty$, then $rP(k) < \pi'_k(\theta, k)$ continues to hold as k decreases, and we may extend O downwards to the boundary of $\mathcal{A}$. This proves the claim.

Let $\tilde{G}(\theta, k)$ be the value of a strategy where, if starting at $(\theta, k) \in O$, the investment is delayed until τ_O where τ_O denote the first exit time of O. Then by Dynkin's formula

$$
\begin{aligned}
\tilde{G}(\theta, k) &= E^{\theta,k}\left[\textstyle\int_0^{\tau_O} \pi(\Theta_t, k)e^{-rt}dt + G(\Theta_{\tau_O}, k)e^{-r\tau_O}\right] \\
&= G(\theta, k) + E^{\theta,k}\left[\textstyle\int_0^{\tau_O}(\pi(\Theta_t, k) - \hat{L}G(\Theta_t, k))e^{-rt}dt\right].
\end{aligned}
$$

where $\hat{L}$ is

$$
\hat{L} = -r + \alpha\theta\frac{\partial}{\partial\theta} + \frac{1}{2}(\beta\theta)^2\frac{\partial^2}{\partial\theta^2}
$$

Remember that $((\theta, k) \in O)$

$$
\begin{aligned}
\hat{L}G(\theta, k) = \tilde{\pi}(\theta, k) &= \pi(\theta, \phi(\theta)) + r\textstyle\int_{\phi(\theta)}^{k} P(x)\,dx \\
&< \pi(\theta, \phi(\theta)) + \textstyle\int_{\phi(\theta)}^{k} \pi'_k(\theta, x)\,dx = \pi(\theta, k)
\end{aligned}
$$

We conclude that $\tilde{G} > G$ in O, contradicting the optimality of G. Thus we conclude that $rP(k) \geq \pi'_k(\theta, k)$ in $\mathcal{A}$.

To prove that the equality must be strict, suppose otherwise that $rP(k) = \pi'_k(\psi(k), k)$. According to the observations above, we may change π for $\theta \leq \psi$ without altering the solution, thus let $\pi'_k(\theta, k) = rP(k)$ for all $(\theta, k) \in \mathcal{A}$. In this case the benefits from the environment will always be at least as high as the interest payment on the value of the extracted resource. Moreover, since $\pi''_{\theta k} \geq 0$, and π'_k is not independent of θ, the value of the environment will exceed the return to the resource value for some values of θ. Thus, in no case will the returns from the environment be less than that from the value of the resource, while in some cases it exceed it. This contradicts the optimality of extraction at $\theta = \psi(k)$. $\square$

Proof. of proposition 2: We first prove that the reservation price is increasing in ψ.

$$\begin{aligned} \tfrac{dQ}{d\psi}(\psi,k) &= \tfrac{\gamma_1}{r}\left[\gamma_1\psi^{\gamma_1-1}\textstyle\int_\psi^\infty \frac{\pi_k'(s,k)}{s^{\gamma_1+1}}ds - \psi^{\gamma_1}\frac{\pi_k'(\psi,k)}{\psi^{\gamma_1+1}}\right] \\ &\geq \tfrac{\gamma_1}{r}\pi_k'(\psi,k)\left[\gamma_1\psi^{\gamma_1-1}\textstyle\int_\psi^\infty \frac{1}{s^{\gamma_1+1}}ds - \psi^{-1}\right] \\ &= 0 \end{aligned}$$

with a strict inequality unless $\pi_k'(\theta,k)$ is constant for $\theta \geq \psi$.

The expected benefits starting at an arbitrary state θ and keeping the capital intact can be found from Lemma 1 of Kobila (1991). Differentiating with respect to k we find the value of a marginal unit of capital.

$$\zeta_k'(\theta,k) = \frac{2}{(\gamma_1-\gamma_2)\beta^2}\left[\theta^{\gamma_1}\int_\infty^\theta \frac{\pi_k'(s,k)}{s^{\gamma_1+1}}ds + \theta^{\gamma_2}\int_0^\theta \frac{\pi_k'(s,k)}{s^{\gamma_2+1}}ds\right].$$

Note that this is similar to G_k' except that π_k' rather than $rP(k)$ is used in the area $\mathcal{A}$. In particular, at the boundary

$$\zeta_k'(\psi(k),k) - G_k'(\psi(k),k) = \frac{2}{(\gamma_1-\gamma_2)\beta^2}\left[\psi(k)^{\gamma_2}\int_0^{\psi(k)} \frac{\pi_k'(s,k) - rP(k)}{s^{\gamma_2+1}}ds\right] < 0$$

The last inequality follows from the conclusion of the proposition above, that $\pi_k'(\psi(k),k) < rP(k)$. Since $P(k) = G_k'(\psi(k),k)$ the claim follows. $\square$

Proof. of proposition 4: Integrating by part gives the formula

$$rP(k) = \pi_k'(\psi(k),k) + \int_{\psi(k)}^{\infty} \pi_{k\theta}''(s,k)\left(\frac{\psi(k)}{s}\right)^{\gamma_1} ds \tag{A.7}$$

assuming that

$$\lim_{s\to\infty} \frac{\pi_k'(s,k)}{s^{\gamma_1}} = 0 \tag{A.8}$$

The right hand side of (7) is increasing in the variable $\psi(k)$. Since γ_1 is increasing when β is decreasing, this implies that $\psi(k)$ increases when β decreases.

To generalize to the case when (8) is not satisfied, note that if $\frac{\pi_k'(s,k)}{s^{\gamma_1}} \geq a > 0$, the integral in (3) will diverge. Hence $\psi(k)$ is well define only if there is a sequence $s_n \to \infty$ such that $\frac{\pi_k'(s_s,k)}{s_n^{\gamma_1}} \to 0$. It follows that the second term in (7) must converge too, i.e. that

$$\int_{\psi(k)}^{s_n} \pi_{k\theta}''(s,k)(\frac{\psi(k)}{s})^{\gamma_1} ds \to 0,$$

and that (7) is satisfied using the limit of this sequence. The rest of the proof is as above. $\square$

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