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Discrete and continuous choice, max-stable processes and independence from irrelevant attributes

by

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Abstract

The Generalized Extreme Value Model was developed by McFadden for the case with discrete choice sets. The present paper extends this model to cases with both discrete and continuous choice sets and choice sets that are unobservable relative to the analyst. We also propose behavioral assumptions that justify random utility functions (processes) that have a max-stable structure i.e., utility processes where the finite dimensional distributions are of the multivariate extreme value type.

Finally we derive non-parametrically testable implications for the choice probabilities in the continuous case.

Keywords: Choice of attributes, random utilities, stochastic demand functions, latent choice sets, IIA, max-stable processes.

1. INTRODUCTION

This paper discusses a particular random utility framework for modeling discrete and continuous choice. We demonstrate how ideas related to the literature of discrete choice models can be exploited and extended to establish a unified framework for discrete and continuous choice.

In the existing consumer demand literature, the stochastic properties of the demand function are usually specified ad hoc. The present approach, however, employes ideas developed, in part, by psychologists to obtain a theoretical basis for the choice of functional form. Examples of this tradition are found in Halldin (1974) and in Suppes et al. (1989), chapter 17. Specifically, we consider choice settings where each alternative is identified by a consumption bundle and a vector of qualitative variables called attributes. The set of attributes that are feasible to the agent is generated by a random device. The agent's preferences are represented by a utility function that is random. The stochastic properties of the demand function are derived from behavioral axioms, of which the most important one is analogous to the "Independence from Irrelevant Alternatives" axiom (IIA), proposed by Luce (1959). Our version of IIA states that, conditional on alternatives with a given level of the consumption bundle, the utilities for attributes are distributed so as to yield (conditional) choice probabilities that satisfy a continuous version of IIA. In the context of discrete choice, it is well known that IIA corresponds to utility functions that are extreme value distributed. We extend this result in the sense that when utility is viewed as a stochastic process with the consumption bundle as parameter, our version of IIA is shown to imply that the utility process is max-stable. This means that the joint distribution of a vector of utilities, evaluated at different consumption bundles, is of the multivariate extreme value type. Once this has been established it is possible to draw on recent developments in probability theory (cf. de Haan, (1984)) to characterize the corresponding probability model for the agent's choices.

A random formulation of the utility function is usually motivated by the econometrician's need to account for unobservable tasteshifters that are assumed to be perfectly foreseeable to the agent. However, in the field of psychology, beginning with

Thurstone (1927), there has been a tradition of interpreting the utility function as random to the agent himself. The justification for this is that in laboratory experiments, individuals have been found to make different decisions under identical experimental conditions. One explanation for this is that the agent's psychological state of mind fluctuates from one moment to the next in a manner that is unpredictable to him. Alternatively, the agent is viewed as having difficulties with the evaluation of the rank order of the alternatives. The framework developed in this paper allows for both interpretations.

A second purpose of this paper is to demonstrate that the proposed framework is able to accommodate choice situations in which the choice sets are latent and vary across agents. This is of considerable interest in many empirical applications where the analyst cannot observe each agent's choice set, but can only observe attributes of the chosen alternative. The choice of geographical location and housing type (where the feasible sites and feasible housing categories are typically unobserved) is one example of this type. Within the conventional framework, it may be rather difficult to account for latent choice sets, and only special cases have been considered in the literature (cf. McFadden, (1981), Thompson, (1989) and Poirier, (1980)).

The application of max-stable processes in the context of random utility models is also discussed by Cosslett (1988) and Resnick and Roy (1991). Their point of departure is the postulation of an upper semicontinuous utility function, defined on a continuous set. Furthermore, they assume that this utility function is a max-stable stochastic process, and they demonstrate that the choice probabilities can be represented through a social surplus function analogous to the discrete case, cf. McFadden (1981). Cosslett (1988) also considers statistical inference in such models. In contrast, we build the theory from axioms on the distribution of individual preferences for attributes, as described above.

The paper is organized as follows. In the next section, we present the particular choice setting and we postulate the behavioral axioms. In Section 3 the choice probabilities are derived for the general case. A non-parametrically testable property, which is analogous to IIA, is also derived. In Section 4, we discuss a special case of the general model framework that is consistent with a continuous version of the Luce model, and finally, in Section 5 we briefly consider the case where the alternatives are discrete.

2. THE CHOICE SETTING AND MAX-STABLE UTILITY PROCESSES

In this paper, except Section 5, it is assumed that each choice alternative is identified by a pair, (x,T), that belongs to $R_+^m \times Y$, $Y \subset R^n$, where $R_+^m \times Y$ is called the choice universe. Here x is a consumption bundle, $R_+^m = \{x \in R^m : x \geq 0\}$, and T is a vector of variables called attributes. As usual, the vector inequality $y>t$ is defined by $(y_1>t_1,\ y_2>t_2,..,y_n>t_n)$. These attributes are assumed to capture qualitative aspects of the alternatives. The agent is assumed to have preferences over $R_+^m \times Y$. The agent's choice set is specified by an economic budget constraint, and possibly additional quantity constraints. The quantity constraints associated with x are specified by

(2.1)
$$x \in K \in \mathcal{B}_+^m,$$

where K is a closed set that is observable and $\mathcal{B}_+^m$ is the Borel field associated with R_+^m. The set of feasible attributes is specified as

(2.2)
$$T \in \mathfrak{J} \cap D,\ \mathfrak{J} \subset Y,\ D \in \mathcal{B}^*,$$

where $\mathfrak{J}$ is a countable, unobservable (to the econometrician), and agent-specific set, and D is an observable set. $\mathcal{B}^*$ is the Borel field associated with Y. The set D is introduced in addition to $\mathfrak{J}$, to allow the analyst to take into account both unobservable and observable restrictions on the set of feasible attributes. This may be desirable in some applications. Since $\mathfrak{J}$ is countable it can be written as an enumeration, $\mathfrak{J}=\{T(z),\ z \in Z\}$, where Z is the set of integers. Since $\mathfrak{J}$ is agent specific, this enumeration is agent specific. Thus, agents are allowed to be heterogeneous with respect to feasible attributes (opportunities). However, for notational simplicity, the agent's index is suppressed.

The set $\Im$ can also be given an alternative and subjective interpretation that is consistent with psychophysical theories of perception. Specifically, we may think of Y as a set of stimuli - or signals, that represent different (latent) qualities or aspects that are relevant to the agent for evaluating the utility of x. While Y is presented to the agent only the subset $\Im$ is used by the agent in his decision making process. The agent's information set $\Im$ varies across identical choice experiments due to psychological processes that are not fully understood. Consequently, $\Im$ in this context is perceived as random to both the agent and the econometrician. The present paper will, however, focus on the interpretation that $\Im \cap D$ is the choice set presented to the agent.

Let $\tilde{R}_+^m = \{x \in R^m : x > 0\}$. The agent's economic budget constraint conditional on attribute vector T(z), is given by

$$p'x \leq f(T(z)), \quad x \in R_+^m, \tag{2.3}$$

where $p \in \tilde{R}_+^m$ is a vector of prices and f(T(z)) is the agent's income net of fixed cost associated with attribute vector T(z).

For example, when we consider the joint choice of geographical location and consumption, x is a vector of goods and T(z) may be the coordinates of location z and f(T(z)) the income minus the fixed cost associated with choosing location z.

In general, the prices may also be attribute specific. We shall, however, not include this case in the general analysis. We demonstrate in examples below how the analysis can be modified to account for prices that depend on attributes.

The agent's preferences are represented by a utility function U(x, z) = u(x,T(z), E(z)) where, as above, z indexes the attributes. Here u(·) is a (deterministic) function, $u: R_+^m \times Y \times R \rightarrow R$, that may depend on observable characteristics of the agent, and E(z) is a tasteshifter associated with attribute vector T(z). The tasteshifters are also agent-specific so that if two different agents have a particular T(z) in their choice sets the corresponding tasteshifters are not necessarily equal. These tasteshifters account for unobserved heterogeneity in tastes across agents and across different attributes for a given agent.

As mentioned above, psychologists interpret the tasteshifters as random relative to the agent in the sense that his preferences for a specific attribute vary from one moment to the next in an unpredictable manner due to instability in the agent's wims, moods and perceptions.

The agent's objective is to maximize utility subject to (2.1), (2.2) and (2.3). Let $x^*(p, C)$ and $T^*(p, C)$ denote the value of x and the attribute that maximize utility, respectively, where $C=K\times D$. In general, since there is no guaranty that this utility maximization problem yields a unique solution, $(x^*(p, C), T^*(p, C))$, we need to impose restrictions on the utility function and the choice set C. The structure of the utility function will be characterized on the basis of a set of assumptions which we shall introduce below.

ASSUMPTION 1: *The function $u(\cdot)$ has the structure $u(x, t, \varepsilon) = v(x, t) + \varepsilon$, where v: $R_+^m\times Y\to R$ is jointly measurable.*

ASSUMPTION 2: *$\{(T(z), \varepsilon(z)), z\in Z\}$ are the points of a Poisson process on $Y\times R$ with intensity measure $\mu G(dt)\cdot M(d\varepsilon)$, where $\mu>0$ is a constant, $G(\cdot)$ is a cumulative distribution function,*

$$0 < \int_y^\infty M(d\varepsilon) < \infty, \quad y\in R,$$

and the mapping

$$y \to \int_y^\infty M(d\varepsilon)$$

is continuous. The mapping $f : Y \to R_+$ is measurable.

Assumption 2 implies that the attributes of $\mathfrak{J}$ are realizations of a Poisson process.

The multiplicative form of the intensity measure means that the tasteshifters and the attributes are independently distributed. In other words, tastes are not correlated with the attribute values.

Recall that a Poisson process on Y×R is completely analogous to a Poisson process on R. Here the realizations occur independently and have coordinates (T(z), E(z)), respectively. The probability that there is a point within

$$(t, t+\Delta t) \times (\varepsilon, \varepsilon + \Delta\varepsilon)$$

is equal to

$$\mu G(\Delta t) M(\Delta\varepsilon) + o(\Delta t \Delta\varepsilon)$$

and the expected number of points within an area, $B \in \mathfrak{B}^* \times \mathfrak{B}$, is given by

$$\Lambda(B) = \int_B \mu G(dt) M(d\varepsilon),$$

where $\mathfrak{B}$ is the Borel field associated with R. The probability that there are exactly n points within B equals

$$\frac{\Lambda(B)^n}{n!} \exp(-\Lambda(B)). \tag{2.4}$$

The notion of heterogeneity in opportunities is obtained through the assumption that different agents face different and independent copies of the Poisson process. Accordingly, this notion is convenient for modeling choice experiment in which the set of feasible attributes is not observed and may vary across agents. In other words, the set of feasible attributes, $\mathfrak{I}$, is perceived as random by the econometrician because he is ignorant about which values are feasible and which are not.

The density of the attributes in $\mathfrak{I}$ is represented by the intensity measure. Let

$$S(t) = \{(y,\varepsilon):\ \varepsilon > b,\ y \le t,\ y \in Y\}$$

and

$$S(Y) = \{(y,\varepsilon):\ \varepsilon > b,\ y \in Y\}$$

where $t \in Y$ and b is a constant. The expected number of Poisson points in S(t) is given by

$$\Lambda(S(t)) = \int_{S(t)} \mu G(dy) M(d\varepsilon) = \mu G(t) \int_b^\infty M(d\varepsilon).$$

Thus

$$\frac{\Lambda(S(t))}{\Lambda(S(Y))} = G(t) \tag{2.5}$$

which demonstrates that G(t) is consistent with a frequency type interpretation, namely as the ratio of the mean number of feasible attributes in S(t) to the mean number of feasible attributes. Thus, loosely speaking, G(t) is the fraction of feasible attributes that have values less than or equal to t. We shall call G(t) the opportunity distribution, cf. Ben-Akiva (1985).

Let $A \in \mathfrak{Z}^*$ and define $U(x,\emptyset) = -\infty$ and

$$U(x,A) = \sup_{T(z) \in A,\, z \in Z} U(x,z) = \sup_{T(z) \in A,\, z \in Z} (v(x,T(z)) + E(z)).$$

We may interpret U(x,A) as the utility of the "aggregate alternative" $\{(x,T): T \in A \cap \mathfrak{Z}\}$. Since there are, with probability one, only denumerably many points in the Poisson process, U(x,A) is a random variable. It also follows from Assumption 2 that if $A_1, A_2 \in \mathfrak{Z}^*$, $A_1 \cap A_2 = \emptyset$, then $U(x,A_1)$ and $U(x',A_2)$ are stochastically independent for all $x, x' \in R_+^m$.

ASSUMPTION 3 (Independence from irrelevant attributes): *Let*

$$P_x(A;D) = P\{U(x,A\cap D) > U(x,D-(A\cap D))\}$$

for $A\in\mathcal{B}^*$ *and non-empty* $D\in\mathcal{B}^*$, $x\in R_+^m$. *For* $A_1\subset A_2\subset D$, A_1, $A_2\in\mathcal{B}^*$, *the measure M has a structure that yields*

$$P_x(A_1;D) = P_x(A_1;A_2)P_x(A_2;D).$$

Moreover, $P_x(\cdot;Y)$ *is absolutely continuous with respect to the measure* μG *and the corresponding Radon-Nikodym derivative is independent of* μG.

The probability $P_x(A; D)$ has the interpretation as the probability that for fixed x the chosen attribute from D belongs to A. Clearly, Assumption 3 is a version of Luce Axiom: "Independence from irrelevant alternatives" (IIA). Specifically, it states that IIA holds for choice sets of the type $\{x\}\times D$, where x is fixed. The absolute continuity assumption follows from the choice theoretic interpretation: Since the intensity measure $\mu G(dt)\cdot M(d\varepsilon)$ represents the density of the feasible points then if

$$\mu\int_A G(dt) = 0$$

for some $A\in\mathcal{B}^*$, this means that (almost surely μG) the points with $T(z)\in A$ are not feasible. Consequently, the corresponding probability of choosing a point within A must also be zero.

The intuition behind the assumption about the Radon-Nikodym derivative is the following: The Radon-Nikodym derivative may be interpreted as a function solely of the agent's preferences. These preferences should not change when the set of feasible attributes change.

We can now prove the following result.

THEOREM 1: *Assume Assumptions 1 and 2 are satisfied and*

$$M(d\varepsilon) = ke^{-a\varepsilon}d\varepsilon, \tag{2.6}$$

where a and k are arbitrary positive constants. Then the choice probabilities satisfy the version of IIA stated in Assumption 3.

A proof of Theorem 1 is given in the Appendix.

The result of Theorem 1 is analogous to the result of Holman and Marley (1961) (cited in Luce and Suppes, (1965), p. 338) where they demonstrate that a random utility model with independent extreme value distributed utilities satisfy IIA. A natural question to ask next is whether the structure (2.6) is also a necessary condition for Assumption 3 to hold. In the context of discrete random utility models this problem has been analyzed by McFadden (1973) and under more general conditions by Yellott (1977). Specifically, they prove that a random utility model with independent utilities satisfies IIA only if the utilities are extreme value distributed. In the present setting the question is settled in Theorem 2 below. The case considered by McFadden and Yellott is a special case within the general framework considered here and it is obtained by assuming that the measure μG has all mass on a finite set of points.

THEOREM 2: *Assume Assumptions 1, 2 and 3. Then*

$$M(d\varepsilon) = e^{-a\varepsilon}kd\varepsilon$$

where a and k are arbitrary positive constants.

A proof of Theorem 2 is given in the appendix.

REMARK 1: If the additive separability condition in Assumption 1 is replaced by multiplicative separability then it can be demonstrated that (2.6) must be replaced by $M(d\varepsilon)=k\varepsilon^{-a-1}d\varepsilon$, for $\varepsilon>0$ and zero otherwise, where $a>0$, $k>0$.

REMARK 2: There is no loss of generality by setting a=1 and k=1. It will become clear below that this corresponds to dividing the utility function by a and subtracting the utility function by logk. Henceforth, we shall therefore fix a=k=1, unless explicitly stated otherwise.

ASSUMPTION 4: *Provided the mapping*

$$t \rightarrow \hat{v}(p,t,K) \equiv \sup_{p'x\leq f(t),x\in K} v(x,t) \tag{2.7}$$

is measurable for $p\in \tilde{R}^m_+$, $K\in \mathcal{B}^m_+$, *then*

$$\int_Y exp(\hat{v}(p,t,R^m_+))G(dt) < \infty.$$

de Haan (1984) demonstrates that Assumption 4 is necessary to ensure that the utility process $\{U(x,A), x\in R^m_+\}$ remains finite with probability one.

When (2.6) and Assumption 4 hold, and $\hat{v}(p,t,R^m_+)$ is measurable, it is easy to derive the finite-dimensional distributions of $\{U(x,A), x\in R^m_+\}$, treated as a stochastic process indexed by x. de Haan (1984), p. 1195, demonstrates how to compute the finite-dimensional distributions of this process in the case where $u(x,t,\varepsilon) = v(x,t)\varepsilon$. When Assumption 1 holds the derivation is completely analogous and gives

(2.8) $$P\left(\bigcap_{j=1}^{s}\left(U(x_j,A)\le u_j\right)\right) = \exp\left(-\int_A \exp\left[\max_{j\le s}\left(v(x_j,t)-u_j\right)\right]\mu G(dt)\right)$$

for $A \in \mathcal{B}^*$, $x_j \in R_+^m$, $u_j \in R$, $j \le s$. Then by de Haan the structure (2.8) implies that $\{U(x,A), x \in R_+^m\}$ is a max-stable process[2)].

ASSUMPTION 5: *For $t \in Y$, $x \to v(x, t)$, $x \in R_+^m$, is continuous except possibly on a set of G measure zero.*

ASSUMPTION 6 (Acceptance treshold): *The agent only takes into account alternatives that have utility above some finite treshold, c, (say) which may be agent-specific.*

Assumption 6 means that the agent apriori discards alternatives as uninteresting if their utility values are less than some treshold and it implies that the set of feasible alternatives above the treshold is a.s. finite, as demonstrated below. The justification for this assumption is that human beings only seem to have capacity to relate to a finite set of alternatives. Moreover, in many choice settings there seems to be a level of standards which determines what is acceptable or not. If there are no feasible alternatives above the acceptance treshold the experiment is simply terminated with no choice being made.

Assumption 6 is not needed to prove the main results of Section 3. However, it implies a simplification of proofs and more important, it allows the interpretation that agents have an a priori level of standards which the alternatives must meet.

Let

(2.9) $$H = \{(t,\varepsilon): \hat{v}(p,t,K) + \varepsilon > c,\ t \in Y,\ \varepsilon \in R\}$$

where c is a constant and $\hat{v}(p,t,K)$ is given by (2.7). When Assumptions 1 and 5 hold then by Lemma 1 in the appendix, $t \to \hat{v}(p,t,K)$ is measurable. When the mapping $t \to \hat{v}(p,t,K)$

is measurable then H is measurable. If c is the treshold that defines the agent's minimum acceptable utility level then the set H contains all combinations of attributes and tasteshifters that are of interest to him. Thus if (2.6) and Assumptions 4 and 6 hold we have that the expected number of acceptable Poisson points is given by

$$\Lambda_c = \int_H e^{-\epsilon} d\epsilon \mu G(dt)$$

$$= e^{-c}\mu \int_Y \exp(\hat{v}(p,t,K)) G(dt)$$

which is finite. Hence, if N_c is the number of Poisson points in H then it is easily demonstrated that $P(N_c>n) \rightarrow 0$ when $n \rightarrow \infty$. Therefore, since $P(N_c=\infty) \leq P(N_c>n)$ for any finite n we can conclude that N_c is finite with probability one.

Let

$$\hat{V}(p,K) = \sup_{(T(z),E(z))\in H, z\in Z} \ \sup_{p'x \leq f(T(z)), x\in K} (v(x,T(z)) + E(z)).$$

The interpretation of $\hat{V}(p, K)$ is as the constrained indirect utility (constrained to K). Since the number of Poisson points in H is a.s. finite and (T(z), E(z)) does not depend on x we get

$$\hat{V}(p,K) = \sup_{(T(z),E(z))\in H, z\in Z} (\hat{v}(p,T(z),K) + E(z)). \tag{2.10}$$

The interpretation of $\hat{v}(p,T(z), K) + E(z)$ is as the constrained conditional indirect utility given attribute z.

ASSUMPTION 7: *For fixed $t\in Y$, $x \rightarrow v(x, t)$ is strictly quasi-concave and increasing in x except possibly for t element of a set of G measure zero.*

Assumptions 1, 5 and 7 imply that there exists a unique $\hat{x}(p, t, K)$ such that

$$v(\hat{x}(p,t,K),t) = \hat{v}(p,t,K)$$

provided K is a closed and convex set. If Assumptions 5 and 7 hold then if x is not at the boundary of K (corner solution) it has the standard regularity properties (cf. Diewert, 1982). The unconstrained conditional indirect utility, $\hat{v}(p, t, R_+^m)$, also has the standard indirect utility properties.

3. CHOICE PROBABILITIES

We are now ready to study the properties of the probability distribution of $x^*(p,C)$ and $T^*(p,C)$. Recall that $C = K \times D$ represents the observable choice constraints. Let $\Delta = \{x \in K, T(z) \in \mathfrak{I} \cap D$: $(x,T(z))$ maximizes utility s.t. the budget constraint (2.3)$\}$. Thus Δ is the set of solutions to the agent's decision problem.

THEOREM 3: *Assume Assumptions 1, 2, 4, 5, 6 and 7 hold and that K is a closed and convex set. Then Δ is a.s. a measurable singleton.*

The proof of Theorem 3 is given in the appendix.

Thus by Theorem 3 the solution to the utility maximization problem is a.s. unique.

The structure of the problem may be interpreted as a two stage procedure: In stage one the agent maximizes $v(x, t)$ with respect to x for fixed t subject to the budget constraints. The solution is $\hat{x}(p, t, K)$. In stage two $\hat{v}(p, T(z), K) + E(z)$ is maximized subject to (2.2) which yields the solution $T^*(p, C)$.

THEOREM 4: *Assume Assumptions 1 to 6. Let*

$$G^*(t,C) = P(T^*(p,C) \leq t \mid \text{There is a point of the Poisson process in } H \cap (D \times R)).$$

Then

(3.1)
$$G^*(t,C) = \frac{\int_{u \leq t, u \in D} \exp(\hat{v}(p,u,K)) G(du)}{\int_D \exp(\hat{v}(p,u,K)) G(du)}.$$

If the density, $g(\cdot)$, of $G(\cdot)$ exists then the density, $g^*(\cdot)$, of $G^*(\cdot)$ exists and is given by

(3.2)
$$g^*(t,C) = \frac{\exp(\hat{v}(p,t,K)) g(t)}{\int_D \exp(\hat{v}(p,u,K)) g(u) du}.$$

A proof of Theorem 4 is given in the appendix.

Eq. (3.1) expresses the c.d.f. of the chosen attribute. It is interesting to note that (3.1) allows an interpretation as the mean value (with respect to G) of the feasible attributes with attribute (vector) less than or equal to t relative to the mean value of the feasible attributes.

THEOREM 5: *Assume Assumptions 1 to 7. Let $B \subset K$, $B, K \in \mathcal{B}_+^m$, B is compact and K is convex and closed. Define*

$$\Phi(B,C)$$
$$= P(x^*(p,C) \in B \mid \text{There is a point of the Poisson process in } H \cap (D \times R)).$$

Then

$$\Phi(B,C) = \frac{\int_{\Omega(B,K)\cap D} \exp(\hat{v}(p,t,K))G(dt)}{\int_{D} \exp(\hat{v}(p,t,K))G(dt)}, \tag{3.3}$$

where

$$\Omega(B,K) = \{t: \hat{v}(p,t,B) > v(x,t), \forall x \in K-B, p'x = f(t)\}.$$

PROOF: Observe first that

$$\Omega(B,K) = \{t: \hat{x}(p,t,K) \in B\}.$$

Since $t \to \hat{x}(p,t,K)$ is measurable it follows that $\Omega(B,K)$ is measurable when B, $K \in \mathfrak{B}^m_+$. Since $x^*(p, C) = \hat{x}(p,T^*(p, C), K)$ we obtain from Theorem 4 that

$$\Phi(B,C) = \int_{D\cap\Omega(B,K)} G^*(dt,C) \tag{3.4}$$

which yields (3.3). Q.E.D.

Similarly to (3.1), (3.3) can also be interpreted as the mean value of feasible attributes, $T(z) \in D$, for which $\hat{x}(p,T(z),K) \in B$, relative to the mean value of feasible attributes in D.

REMARK: Resnick and Roy (1991), p. 287, prove Theorem 3 and that (3.3) holds without Assumption 6. However, recall that our motivation for introducing Assumption 6 is, in addition to mathematical convenience, to demonstrate that the choice probabilities given by (3.3) are consistent with choice from a finite set of acceptable and feasible attributes.

The next Corollary is immediate.

COROLLARY 1: *Under the assumptions of Theorem 5 the demand function, $x^*(p, C)$, can be expressed as*

$$x^*(p,C) = \hat{x}(p,T^*(p,C),K), \tag{3.5}$$

where for fixed t, $(p, K) \to \hat{x}(p, t, K)$ is a constrained Marshallian demand function and T^* is a random variable with c.d.f. $G^*(\cdot, C)$.

We shall now see that the general framework developed above contains conventional Marshallian demand functions as special cases. To realize this consider the utility specification

$$v(x,T(z))/\sigma + E(z) \tag{3.6}$$

where $\sigma>0$ is a parameter that determines the significance of the tasteshifter E(z) relative to the structural part, v(x,T(z)). Consider first the case when $\sigma \to 0$. Then by means of Theorem 4 it is easily demonstrated that $G^*(\cdot)$ converges to a degenerate density, i.e., $T^*(p,C)$ converges to a deterministic variable that equals $\text{argmax}_{t\in D}\hat{v}(p,t,K)$ for given (p,K). Consequently, by Corollary 1, $x^*(p,C)$ reduces to a deterministic demand function. The intuition is that when σ is small then the influence of $\{E(z)\}$ becomes negligible and the decision problem therefore reduces to solving the maximization problem

$$\sup v(x,T(z)).$$
$$\text{s.t. } p'x \le f(T(z))$$

Consider next the case when $\sigma \to \infty$ and D=Y. Then, according to intuition the tasteshifter E(z) becomes very large relative to the structural term $v(x,T(z))/\sigma$. Therefore, the chosen attribute vector will be determined by the maximization of E(z) s.t. $T(z) \in \mathfrak{Z}$ which implies that $T^*(p,K\times Y)$ tends weakly towards a random variable with c.d.f. G(t).

We prove this result formally in Corollary 2 below.

COROLLARY 2: *Assume that the utility function has the form (3.6), D=Y and that the assumptions of Theorem 6 hold. When $\sigma \to \infty$ then the demand function, $x^*(p,K\times Y)$, tends to a Marshallian type demand function conditional on a random variable, $\tilde{T}$, that has c.d.f. $G(\cdot)$.*

PROOF: Replace $\hat{v}(p,u,K)$ by $\hat{v}(p,u,K)/\sigma$ in (3.1). Then it follows immediately from (3.1) and the Lebesgue Dominating Convergence Theorem that

$$\lim_{\sigma \to \infty} G^*(t, K \times Y) = G(t).$$

By Corollary 1 we thus obtain the desired result. Q.E.D.

Corollary 2 demonstrates that the conventional econometric formulation (cf. Varian, (1984), p.p. 181-187) of Marshallian demand functions in the presence of unobserved heterogeneity in preferences formally has the same structure as the special case considered in Corollary 2. However, while $\tilde{T}$ would be interpreted as a tasteshifter in the conventional formulation it is here interpreted as an exogenously determined choice variable.

In the next corollary we shall consider the case where $x^*(p, C)$ is observed and components of the vector $T^*(p, C)$ also are observed. Thus, we suppose now that

$$T^*(p,C) = (T_1^*(p,C), T_2^*(p,C))$$

where the subvector T_1^* is observed while T_2^* is unobserved. Let $Y=Y_1\times Y_2$ be the corresponding decomposition of Y i.e., T_j^* takes values in Y_j, j = 1, 2. If the mapping

$$t_2 \to \hat{x}(p,t_1,t_2,K),\ t_1 \in Y_1,\ t_2 \in Y_2$$

is invertible for fixed t_1, let

$$x \to \hat{t}_2(p,x,t_1,K),\ x \in K,\ t_1 \in Y_1$$

be the inverse mapping.

For notational simplicity we only state the next corollary for the case when D=Y.

COROLLARY 3: *Assume that $x \to \hat{t}_2(p, x, t_1, K)$ exists and is continuously differentiable for x belonging to the interior of K. Then if the density, g, of G exists the density, $\varphi_1(\cdot, K)$, of $(x^*(p, K \times Y), T_1^*(p, K \times Y))$ exists and is given by*

$$\varphi_1(x,t_1,K) = \tag{3.7}$$

$$\frac{\exp(v(x,t_1,\hat{t}_2(p,x,t_1,K)))g(t_1,\hat{t}_2(p,x,t_1,K))\,|J(p,x,t_1,K)|}{\int_Y \exp(\hat{v}(p,t,K))g(t)dt}$$

where x lies in the interior of K and $J(p,x,t_1,K)$ is the Jacobian of the mapping $t \to (t_1, \hat{t}_2(p,x,t_1,K))$.

PROOF: Obviously, we have

$$\varphi_1(x,t_1,K) = g^*(t_1,\hat{t}_2(p,x,t_1,K),Y)\,|J(p,x,t_1,K)|.$$

The result now follows directly from (3.2). Q.E.D.

An important problem is to obtain non-parametrically testable properties for the demand probabilities. The next theorem proposes such a test.

THEOREM 6: *Let K_1, $K_2 \in \mathcal{B}_+^m$ be convex closed sets and let A_1, $A_2 \in \mathcal{B}_+^m$ be disjoint compact convex sets that belong to the interior of $K_1 \cap K_2$. Under Assumptions 1 to 7 the choice probability measure (3.3) satisfies*

$$\frac{\Phi(A_1, K_1 \times D)}{\Phi(A_2, K_1 \times D)} = \frac{\Phi(A_1, K_2 \times D)}{\Phi(A_2, K_2 \times D)}.$$

for $D \in \mathcal{B}^*$.

PROOF: From Assumption 5 and 7 it is clear that the conditional demand functions, $\hat{x}(p,t,K_j)$, j=1,2, are determined by the usual first order conditions. This means that when $\hat{x}(p,t,K_j)$ lies in the interior of K_j then it coincides with $\hat{x}(p, t, R_+^m)$ (interior solution). But this means that since A_j belongs to the interior of K_j then $t \in \Omega(A_j, K_j)$ implies that

$$\hat{x}(p,t,K_j) = \hat{x}(p,t,R_+^m)$$

and

$$\hat{v}(p,t,K_j) = \hat{v}(p,t,R_+^m).$$

Accordingly, by Theorem 5 we realize that

$$\frac{\Phi(A_1, K_j \times D)}{\Phi(A_2, K_j \times D)}$$

is independent of K_j. This completes the proof. Q.E.D.

A moment's reflection reveals that for IIA to hold at the corners there must be a one-to-one correspondance between points in R_+^m and points in Y. Since this is not the case under the present assumptions, $\Omega(A, K)$ will depend on K when A contains elements of the boundary of K and thus the IIA property may be violated.

EXAMPLE 1 (Demand for housing and consumption): In the present example we assume that the agent faces the choice between different housing opportunities where house z is characterized by two attributes $T(z)=(T_1(z),T_2(z))$. Here $T_2(z)$ denotes the minimum energy use for heating, while $T_1(z)$ denotes the annual user cost. $T_1(z)$ is observable while $T_2(z)$ is unobservable. The agent's utility function is given by a Stone-Geary type utility;

$$U(x,z) = \alpha_1 \log(x_1-\beta) + \alpha_2 \log(x_2-T_2(z)) + E(z) \tag{3.8}$$

where $\alpha_j>0$, $j = 1, 2$, and β are unknown parameters, $E(z)$ is a tasteshifter, x_2 is the consumption of energy related to heating and cooking and x_1 is consumption of other goods. The budget constraint is given by (conditional on house z)

$$p_1 x_1 + p_2 x_2 \leq y - T_1(z), \tag{3.9}$$

where p_1, p_2 are the prices respectively, and y is the agent's income.

The agent's set of feasible housing opportunities is not observable to the analyst and therefore we assume that $\{(T(z), E(z)), z = 1, 2,...\}$ are generated by a Poisson law with intensity measure

$$g(t_1,t_2)dt_1dt_2 \cdot e^{-\varepsilon}d\varepsilon.$$

The interpretation of $g(t_1, t_2)dt_1dt_2$ is as the fraction of houses with attributes $T_1(z)\in(t_1, t_1+dt_1)$, $T_2(z)\in(t_2, t_2+dt_2)$ that are feasible to the agent. If, for example, statistics on the aggregate number of houses that have attributes satisfying $T_1(z)\leq t_1$, $T_2(z)\leq t_2$, exists this could be applied to obtain auxiliary estimates of the opportunity distribution, $G(t_1, t_2)$.

The utility index (3.8) implies the following demand, $\hat{x}(p, t)$ conditional on t:

$$\hat{x}_2(p,t) = \frac{\alpha_2}{\alpha p_2}(y-p_1\beta-t_1) + \frac{\alpha_1 t_2}{\alpha} \tag{3.10a}$$

and

(3.10b) $$\hat{x}_1 p_1 = y - t_1 - p_2\hat{x}_2.$$

where $\alpha_1+\alpha_2=\alpha$. Thus the demand is given by (cf. Corollary 1)

(3.11a) $$x_2^*(p) = \frac{\alpha_2}{\alpha p_2}(y - p_1\beta - T_1^*) + \frac{\alpha_1 T_2^*}{\alpha},$$

and

(3.11b) $$x_1^*(p)p_1 = y - T_1^* - p_2 x_2^*(p).$$

For simplicity, assume that $T_1(z)$ and $T_2(z)$, z=1,2,..., are independently distributed, i.e., $g(t_1, t_2) = g_1(t_1)g_2(t_2)$. From (3.10) and (3.8) it follow that the function $\hat{v}(p, t, R_+^2)$ takes the form

(3.12) $$\hat{v}(p,t,R_+^2) = \alpha\log(y-t_1-p_1\beta-p_2t_2) + \alpha_1\log\left(\frac{\alpha_1}{\alpha p_1}\right) + \alpha_2\log\left(\frac{\alpha_2}{\alpha p_2}\right).$$

The density of (T_1^*,T_2^*) is then obtained from (3.2) and equals

(3.13) $$g^*(t_1,t_2) = \frac{(y-t_1-p_1\beta-p_2t_2)^\alpha g_1(t_1)g_2(t_2)}{\int\int(y-u_1-p_1\beta-p_2u_2)^\alpha g_1(u_1)g_2(u_2)du_1du_2}.$$

In particular, the conditional density of the unobservable T_2^* given T_1^* is equal to

(3.14) $$g_2^*(t_2|t_1) = \frac{(y-t_1-p_1\beta-p_2t_2)^\alpha g_2(t_2)}{\int(y-t_1-p_1\beta-p_2u_2)^\alpha g_2(u_2)du_2}.$$

Thus (3.14) implies a simultaneous equation bias problem that is similar to the selectivity bias problem (see Heckman, (1979)) because the unobservable T_2^* is the outcome of a choice variable which is correlated with the other choice variable T_1^* and (y, p).

From Corollary 3 we obtain that the joint density of (x_2^*, T_1^*) is given by

$$(3.15) \qquad \varphi_1(x_2,t_1) = \frac{(y-p_2x_2-p_1t_1-p_1\beta)^\alpha g(t_1,\hat{t}_2(p,x_2,t_1))}{\iint\limits_{R_+^2}(y-p_2u_2-p_1u_1-p_1\beta)^\alpha g(u_1,\hat{t}_2(p_1,u_2,u_1))du_1du_2}$$

where

$$(3.16) \qquad \hat{t}_2(p,x_2,t_1) = \frac{\alpha x_2p_2-\alpha_2(y-p_1\beta-t_1)}{\alpha_1p_2}.$$

EXAMPLE 2 (Choice and frequency of restaurant visits): In this example the agent's choice set of available restaurants depends on his geographical location.

Let $T_2(z)$ be an attribute that characterizes restaurant z, such as location and category, and let exp $(T_1(z))$ be an indicator of the price level (composite price). Let x_1 be the agent's restaurant consumption and let x_2 be the remaining consumption. We assume that the agent only visits one restaurant. Assume that utility has the form

$$(3.17) \qquad U(x,z) = u(x_1\exp(T_2(z)),x_2) + E(z)$$

with budget constraint

$$x_1\exp(T_1(z)) + x_2 \leq y,$$

where x_2 is taken as the numeraire. The indirect utility that correponds to (3.17) conditional on (T(z), E(z)) has the form

$$(3.18) \qquad \hat{V}(z) \equiv v^*(T_1(z)-T_2(z),y) + E(z),$$

where $v^*(\cdot,\cdot)$ is a function that is decreasing and convex in it's first argument and increasing in the second.

Suppose v^* has the form

(3.19) $$v^*(p,y) = \frac{\theta}{d-1}e^{(1-d)p} - \frac{e^{-\eta y}}{\eta}, \eta \neq 0, \theta > 0,$$

where d, θ and η are parameters (cf. Haneman, 1984, eq. (3.16)). Then expenditure conditional on restaurant z equals

(3.20) $$\hat{x}_1(T(z))\exp(T_1(z)) = \theta \exp((1-d)(T_1(z)-T_2(z))+\eta y).$$

By (3.2) the density of the chosen attributes, (T_1^*, T_2^*), is given by

(3.21) $$g^*(t_1,t_2) = \frac{\exp(b\exp((1-d)(t_1-t_2)))g(t_1,t_2)}{\iint_{R^2}\exp(b\exp((1-d)(u_1-u_2)))g(u_1,u_2)du_1du_2}$$

where $b=\theta/(d-1)$. By applying (3.7) the density of $(\log(x_1^*\exp(T_1^*)), T_1^*)$ is readily demonstrated to be

(3.22) $$\varphi_1(s,t_1) = \frac{\exp(ce^{-\eta y+s})g(t_1,\hat{t}_2(s,t_1))}{\iint_{R^2}\exp(ce^{-\eta y+w})g(u,\hat{t}_2(w,u))dudw}$$

where s is the logarithm of the level of restaurant expenditure,

(3.23) $$\hat{t}_2(s,t_1) = t_1 + \frac{s-\eta y-\log\theta}{d-1}$$

and $c = b/\theta$.

EXAMPLE 3 (Haneman's perfect substitutes model, Haneman, op cit. p.p. 548-552): Assume that there are m different substitutes of a good. The consumption of substitute j is x_j. The remaining consumption of other goods is denoted w. The utility function is

(3.24) $$U(x,w,z) = U^*(\sum_{j=1}^{m} x_j\lambda_j(T(z))\exp(E(z)),w),$$

$x_j \geq 0$, $w \geq 0$, where U^* is a conventional utility function,

$$\lambda_j(t) = \begin{cases} \lambda(t) & \text{for } t\in D_j \\ 0 & \text{otherwise,} \end{cases}$$

for some positive function $\lambda(\cdot)$ and $\{D_j\}$ is a partition of Y, $D_j \in \mathcal{B}^*$. This means that the functions $t \rightarrow \lambda_j(t)$, j=1,2,...,m, have disjoint supports. The budget constraint is given by

$$w + p'x \leq y$$

and the variables $\{(T(z), E(z)), z = 1, 2,...\}$ are the points of the Poisson process as described above.

We may interpret the set D_j as the set of j-specific quality attributes. The function $\lambda_j(T(z))$ modifies the utility of x_j according to unobservable quality aspects. The term $\exp(E(z))$ represents, as above, the influence of heterogeneity in tastes relative to the "objective" quality attributes.

Following Haneman (1984) the utility structure above implies that, conditional on z, the consumer will choose a "corner solution" in that for some j, $x_j>0$ and $x_i=0$ for all $i\neq j$. The corresponding indirect utility conditional on z and $x_j>0$ equals

(3.25) $$v^*(p_j\exp(-E(z))/\lambda_j(T(z)),y), \quad T(z)\in D_j,$$

where v^* is a function that is decreasing and convex in its first argument and increasing in the second. Therefore the indirect utility conditional on $x_j>0$ is given by

(3.26) $$v^*(p_j/\psi_j, w)$$

where

$$\psi_j = \sup_{T(z)\in D_j} \lambda(T(z))\exp(E(z)). \tag{3.27}$$

Since D_j, j=1,2,...,m, are disjoint it follows that ψ_j, j=1,2,...,m, are independent. Moreover, by (2.8), $\log\psi_j$ is type III extreme value distributed so that we can write

$$\psi_j = \exp(\alpha_j + E_j), \tag{3.28}$$

where E_j, j=1,2,...,m, are i.i.d. with distribution function $\exp(-e^{-\varepsilon_j})$ and α_j is a parameter. Similarly the demand function given $x_j>0$ can be expressed as

$$\bar{x}_j \equiv h(p_j/\psi_j, w)\psi_j \tag{3.29}$$

where h is the demand function that corresponds to maximizing $U^*(x_j, y)$ subject to $p_j x_j + y \leq w$. Thus we have obtained the same model as Haneman, op.cit. (cf. Haneman, eqs. (3.5) and (3.6)).

EXAMPLE 4 (Consumer demand where the products have different qualities): This example is similar to example 2 and 3. The agent's choice problem is to choose between different qualities of a differentiated product and how much to consume. There are two products (goods). The quality of the j-th good is measured by $T_j(z)$, j=1,2, where z indexes the different variants. The utility function is

$$U(x,z) = \alpha_1 \log(x_1 T_1(z) - \beta_1) + \alpha_2 \log(x_2 T_2(z) - \beta_2) + E(z) \tag{3.30}$$

where x_j denotes the consumption of good j and α_1, α_2, β_1, β_2 are positive parameters. We assume that the consumer only buy one quality variant at a time. A justification for this may be that the consumer only shops at one market place at a time.

The budget constraint is given by

(3.31) $$x_1 Q_1(z) + x_2 Q_2(z) \le y$$

where $Q_j(z)$ is the price of good j with quality $T_j(z)$.

The set of feasible quality attributes, prices and tasteshifters, $\{(T_1(z), T_2(z), Q_1(z), Q_2(z), E(z)), z=1,2,...\}$ is generated by a Poisson law with intensity measure

$$\mu g(t,q)\,dt\,dq \cdot e^{-\varepsilon}d\varepsilon \equiv \mu g(t_1,t_2,q_1,q_2)\,dt_1 dt_2 dq_1 dq_2 \cdot e^{-\varepsilon}d\varepsilon,$$

where g(t,q) is a probability density. The interpretation of g(t,q) is as the density of variants in the market with given levels of price and quality.

Let $R_j(z) = Q_j(z)/T_j(z)$. Then the consumer's maximization problem is equivalent to maximizing

(3.32) $$\tilde{U}(w,z) = \alpha_1 \log(w_1 - \beta_1) + \alpha_2 \log(w_2 - \beta_2) + E(z),$$

(3.33) $$\text{s.t.} \quad w_1 R_1(z) + w_2 R_2(z) \le y.$$

Conditional on quality, the expenditure functions are given by

(3.34a) $$\hat{w}_1(r) r_1 = \hat{x}_1(t,q) q_1 = \beta_1 r_1 + \frac{\alpha_1}{\alpha}(y - \beta_1 r_1 - \beta_2 r_2)$$

and

(3.34b) $$\hat{w}_2(r) r_2 = \hat{x}_2(t,q) q_2 = \beta_2 r_2 + \frac{\alpha_2}{\alpha}(y - \beta_1 r_1 - \beta_2 r_2),$$

where $r_j = q_j/t_j$ and $r = (r_1,r_2)$. The corresponding indirect utility is given by

(3.35) $$\hat{v}(t,q,R_+^m) = \alpha \log(y - \beta_1 r_1 - \beta_2 r_2) - \alpha_1 \log r_1 - \alpha_2 \log r_2 + k$$

where k is a constant and $\alpha = \alpha_1 + \alpha_2$. Let $g^*(r,q)$ be the joint density of the choosen attributes, (R_1^*,R_2^*), and prices. From Theorem 4 we get

(3.36)
$$g^*(r,q) = \frac{(y-r_1\beta_1-r_2\beta_2)^{\alpha} r_1^{-\alpha_1} r_2^{-\alpha_2} \tilde{g}(r,q)}{\iint (y-u\beta_1-v\beta_2)^{\alpha} u^{-\alpha_1} v^{-\alpha_2} \tilde{g}(u,v)\,du\,dv},$$

where $u = (u_1,u_2)$, $v = (v_1,v_2)$ and $\tilde{g}(r,q)$ is the density of the attributes and prices, $(R_1(z), R_2(z), Q_1(z), Q_2(z))$, and it is given by

(3.37)
$$\tilde{g}(r,q) = g\left(\frac{q_1}{r_1},\frac{q_2}{r_2},q_1,q_2\right) q_1 q_2 r_1^{-2} r_2^{-2}.$$

4. A PURE CHOICE-OF-ATTRIBUTE MODEL WITH RANDOM CHOICE SETS

In this section we consider choice settings that consist of only qualitative alternatives with attributes $T(z)\in \mathfrak{I}\cap D$, $D\in \mathfrak{Z}^*$. Thus the set of feasible attributes is associated with realizations of a Poisson process that have points with utility

(4.1)
$$U(z) = u(T(z)) + E(z),$$

where $\{T(z), E(z)\}$ are the points of the Poisson process on $Y\times R$ with intensity measure

$$\mu G(dt) \cdot e^{-\varepsilon} d\varepsilon.$$

The choice probability measure is now defined by

$$\psi(B,D) \tag{4.2}$$
$$= \Big(\sup_{T(z)\in B,(T(z),E(z))\in H, z\in Z} (u(T(z))+E(z)) > \sup_{T(z)\in D,(T(z),E(z))\in H, z\in Z} (u(T(z))+E(z)) \,\Big|$$
$$\text{There is a point of the Poisson process in } H\cap(D\times R)\Big)$$

where $B \subset D$, $B, D \in \mathcal{B}^*$ and $H = \{(t, \varepsilon): u(t) + \varepsilon > c, t\in Y, \varepsilon\in R\}$.

THEOREM 7: *Assume that the utility function has the structure (4.1) and that Assumption 6 holds. Assume also that*

$$\int_Y exp(u(t))G(dt) < \infty.$$

Then the choice probability measure is given by

$$\psi(B,D) = \frac{\int_B exp(u(t))G(dt)}{\int_D exp(u(t))G(dt)}. \tag{4.3}$$

where $B \subset D$, $D, B \in \mathcal{B}^*$.

PROOF: Since the points of the Poisson process are independently distributed it follows that

$$\sup_{T(z)\in B,(T(z),E(z))\in H, z\in Z} U(z) \quad \text{and} \quad \sup_{T(z)\in D-B,(T(z),E(z))\in H, z\in Z} U(x)$$

are independently distributed. Similarly to the proof of Theorem 4 it follows that for any $A \in \mathcal{B}^*$

$$(4.4) \qquad P\left(\sup_{T(z)\in A,(T(z),E(z))\in H,z\in Z} U(z)\leq u\right) = \begin{cases} \exp(-e^{-u}\int_A \exp(u(t))\mu G(dt)) & \text{for } u\geq c, \\ 0 & \text{for } u<c. \end{cases}$$

From (4.4), (4.3) now follows by straight forward calculus. Q.E.D.

The next corollary follows immediately.

COROLLARY 4: *The choice probability measure (4.3) satisfies IIA, i.e.,*

$$\frac{\psi(B_1,D_1)}{\psi(B_2,D_1)} = \frac{\psi(B_1,D_2)}{\psi(B_2,D_2)}$$

for $B_1, B_2 \in D_1 \cap D_2$, $D_j, B_j \in \mathcal{B}^*$, $j = 1, 2$.

The model $\psi(B, D)$ may be called the continuous Luce model because it is consistent with IIA for continuous choice sets (see McFadden, (1976)). It allows, however, a more general interpretation of the choice environment than the Luce model in that it explicitly accounts for latent opportunity sets that vary across agents. Alternatively, as discussed in Section 2, the choice set, $\mathfrak{I}\cap D$, may be interpreted as random to the agent himself.

Ben-Akiva et al. (1985) have also developed a continuous Luce (logit) model with latent opportunities of the same form as $\psi(B, D)$. Their model is obtained by starting from a discrete choice set model and letting the set of feasible attributes converge to a continuous set. Our results demonstrate that the limiting continuous model is in fact consistent with a particular representation of the preferences.

EXAMPLE 5 (A disequilibrium model for labor supply): A simplified version of this model assumes that the agent chooses from a latent set of feasible hours-wage packages, T(z)=(H(z),W(z)), where H(z) and W(z) are the hours and wage of job z. It is also assumed that given a job then hours of work associated with the job is given. For simplicity we only consider the choice of job given that the agent wishes to work and jobs are available. The utility function has the structure

$$U(h,C,z) = v^*(h,C) + E(z), \tag{4.5}$$

where h denotes hours and C is disposable income. The variable E(z) is a tasteshifter that accounts for non-pecuniary aspects of job z and v^* is a function that is concave, increasing in C and decreasing in h. For a given job, z, (say) the budget constraints are

$$h = H(z)$$

$$C = C(z) = W(z)H(z) + I - \gamma(W(z)H(z),I) \tag{4.6}$$

where $\gamma(\cdot)$ is the tax function and I is non-labor income.

The set of hours-wage packages that are feasible to the agent is not observed and thus $\{(H(z), W(z), E(z)), z = 1, 2,...\}$ are assumed to be the points of a Poisson process with intensity measure

$$\mu g(h,w)dhdw \cdot e^{-\epsilon}d\epsilon,$$

where g(h,w) corresponds to the density of G(t) in Theorem 7.

The interpretation of g(h,w)dhdw is as the fraction of jobs with $H(z)\in(h, h+dh)$, $W(z)\in(w,w+dw)$ that are feasible to the agent. Let

$$v(h,w) = v^*(h,hw+I-\gamma(hw,I)). \tag{4.7}$$

When the budget constraints are inserted in (4.5) we get

(4.8) $$U(H(z),W(z),z) = v(H(z),W(z)) + E(z).$$

Let (H^*,W^*) be the hours and wage of the chosen job. From (4.3) we get the density, $\psi(h,w)$, of (H^*,W^*):

(4.9) $$\psi(h,w) = \frac{\exp(v(h,w))g(h,w)}{\iint \exp(v(x,y))g(x,y)dxdy}.$$

Dagsvik and Strøm (1992) have applied this approach in empirical analyses of labor supply with taxes in Norway.

5. DISCRETE CHOICE

The type of generalized extreme value models (GEV) that have been applied in empirical work in discrete choice problems are mostly the logit and the nested logit model. McFadden (1981) demonstrates how GEV choice probabilities can be expressed by means of the social surplus function. Apart from the logit and nested logit case it is not obvious how to specify empirically tractable and theoretically justified specifications of the social surplus function. As is wellknown, the nested logit model is, as pointed out by several authors, not always ideal (see McFadden, 1981). The nested logit model pressumes that the choice set can be organized according to a tree-structure which is not always a natural a priori assumption.

The formulation by means of the max-spectral representation offers an appealing alternative approach for the specification of flexible parametric forms of choice probabilities within the GEV class. Furthermore, in many applications the spectral representation may be a plausible formulation for theoretical reasons, cf. the discussion in Section 2. Let us briefly consider this alternative below. Let

(5.1) $$U(j,z) = v(j,T(z)) + E(z)$$

where j = 1, 2,...,m, is an indexation of the discrete alternatives and z indexes a countable set of unobservable alternatives characterized by attribute T(z). Thus the setup is completely analogous to the one in Section 2 with x replaced by a discrete index, j. The budget constraint is replaced by

$$j \in K \tag{5.2}$$

where K is a subset of {1,2,...,m}. For simplicity we only consider the case where the chosen attribute, T^*, is unobservable. As above {(T(z), E(z)), z = 1, 2,...} are the points of the Poisson process on Y×R with intensity measure

$$\mu G(dt)e^{-\varepsilon}d\varepsilon.$$

Let

$$U_j = \sup_{z \in Z} U(j,z).$$

Then $(U_1, U_2,...,U_m)$ is multivariate extreme value distributed (type III) and accordingly the choice model belongs to McFadden's GEV class (cf. McFadden, (1981)).

Of course, it is not necessary to require that the coordinates {T(z)} of the Poisson points can be interpreted as attributes of latent choice alternatives to apply the spectral representation framework. However, if such an interpretation applies then it provides a natural justification for the max-stable framework.

In many empirical analyses of discrete choice there are observable attributes associated with the discrete alternatives and observable variables that characterize the agent. Thus a typical characterization of the spectral function, v(j,t), may be

$$v(j,t) = h(Q_j, s, \theta, t), \tag{5.3}$$

where Q_j is a vector of observed variables that characterizes alternative j, θ is an unknown parameter vector and h(·) is a suitably chosen function.

As before define

$$\hat{v}(t,K) = \max_{j\in K} v(j,t). \tag{5.4}$$

Let J(K) denote the selected discrete alternative from K determined by utility maximazation and let φ(j, K) be the corresponding choice probability, i.e.,

$$\sup_{z\in Z}(v(J(K),T(z)) + E(z)) \equiv \sup_{z\in Z}(\hat{v}(K,T(z)) + E(z)) \tag{5.5}$$

and

$$\varphi(j,K) \equiv P(J(K)=j) = P(\sup_{z\in Z} U(j,z) = \max_{i\in K}\sup_{z\in Z} U(i,z)). \tag{5.6}$$

Then, similarly to (3.3)

$$\varphi(j,K) = \frac{\int_{\Omega(j,K)} \exp(v(j,t))G(dt)}{\sum_{i\in K}\int_{\Omega(i,K)} \exp(v(i,t))G(dt)}, \quad j\in K, \tag{5.7}$$

where $K \subset \{1, 2,...,m\}$ and

$$\Omega(i,K) = \{t: v(i,t) > v(k,t), \forall k\in K-\{i\}, i\in K\}.$$

The proof of (5.7) is completely analogous to the proof of (3.3).

As in the continuous case we realize that the choice probability (5.7) will not in general satisfy IIA. A sufficient condition for IIA to hold is that the spectral functions have disjoint supports because then the utilities $U_1,U_2,...,U_m$, become independent, cf. de Haan (1984). However, this condition is not necessary. Strauss (1979) has demonstrated that it is possible to specify GEV models with interdependent utility functions that satisfy IIA.

EXAMPLE 6 (Discrete choice): Let Y=R and

$$v(j,t) = \alpha Q_{1j} + \sigma t Q_{2j}, \quad j \leq m,$$

where (Q_{1j}, Q_{2j}) are observable attributes of alternative j and α and σ are parameters. Then

$$\Omega(j,K) = \{t: \alpha Q_{1j} + \sigma t Q_{2j} > \max_{i \in K-\{j\}} (\alpha Q_{1i} + \sigma t Q_{2i}), j \in K\}.$$

If g is the standard normal density it follows from (5.7) that the choice probability density is given by

$$\varphi(j,K) = \frac{\int_{\Omega(j,K)} \exp(\alpha Q_{1j} + \sigma Q_{2j} t - \frac{1}{2} t^2) dt}{\sum_{i \in K} \int_{\Omega(i,K)} \exp(\alpha Q_{1i} + \sigma Q_{2i} t - \frac{1}{2} t^2) dt}. \tag{5.8}$$

6. CONCLUSIONS

The main focus in the present paper has been to develop a theoretical foundation for consumer demand when the choice variables are partly qualitative and belong to a latent choice set. Specifically, the point of departure is a choice setting where the agent is assumed to have preferences over choice alternatives where each alternative is identified by a consumption bundle and a vector of (qualitative) variables called attributes. The main axiom that is postulated states that, conditional on alternatives with a given level of the consumption bundle, the preferences over attributes are distributed so as to yield corresponding (conditional) choice probabilities that satisfy a continuous version of IIA. It follows that under suitable regularity conditions this axiom implies that the agent's utility function of the consumption bundle belongs to the class of max-stable stochastic processes. The corresponding distribution function for the demand is derived and a non-parametrically testable property is obtained. Furthermore, a continuous version of the Luce model is obtained as a special case.

Several examples are discussed in order to clarify the interpretation and the potential of the framework for applications within different fields.

Finally, we discuss discrete choice models with reference to de Haan's (1984) spectral representation result for max-stable processes.

Appendix

PROOF OF THEOREM 1: Let $A \in \mathfrak{B}^*$ and $M(d\varepsilon) = ke^{-a\varepsilon}d\varepsilon$. From Assumptions 1 and 2 we get

(A.1) $\quad P(U(x,A) \leq u)$

$$= P(\text{There are no points of the Poisson process in the set } \{(t,\varepsilon):(t,\varepsilon) \in A \times R, v(x,t) + \varepsilon > u\})$$

$$= \exp\left(-re^{-au} \int_{t \in A} \exp(av(x,t))\, G(dt)\right),$$

where $r=k\mu/a$. But (A.1) demonstrates that U(x,A) is type III extreme value distributed. Moreover, since the Poisson points are independently distributed U(x,D) and U(x,D-A) are independent for $A \subset D$, $A,D \in \mathfrak{B}^*$. Thus

$$\text{(A.2)} \qquad P_x(A;D) = P(U(x,A) > U(x,D-A)) = \frac{\int_{t \in A} \exp(av(x,t))\, G(dt)}{\int_{t \in D} \exp(av(x,t))\, G(dt)},$$

which we realize is a continuous version of the Luce model. From (A.2) it follows immediately that Assumption 3 holds. Q.E.D.

PROOF OF THEOREM 2: Let $D \in \mathfrak{B}^*$ and choose a finite partition $\{D_j\}$ of Y such that either $D_j \cap D = \emptyset$ or $D_j \cap D = D_j$, i.e., the D_j's are either within D or outside D. By Assumption 3

$$P_x(D_j;D) = \frac{P_x(D_j;Y)}{\sum_i P_x(D_i;Y)}.$$

By Assumption 2 and from Yellott (1977), p.p. 134-135, it follows that

$$U(x,D_j) \overset{d}{=} \log P_x(D_j; Y) + \eta_j, \tag{A.3}$$

where $\overset{d}{=}$ means equality in distribution and $\eta_j = \eta(x, D_j)$, j=1,2,..., are independent random variables with distribution

$$P(\eta_j \le \eta) = \exp(-e^{-a\eta}),$$

where a>0 is a constant. Without loss of generality we may choose a=1.

Now since $P_x(\cdot;Y)$ is absolutely continuous with respect to μG we have that

$$P_x(D_j; Y) = \int_{D_j} v^*(x,t)\mu G(dt)$$

for some t-measurable function $v^* : (x, t) \rightarrow v^*(x, t) > 0$. Thus (A.3) implies that for $A \in \mathcal{B}^*$

$$P(U(x,A) \le u) = \exp(-e^{-u}\int_A v^*(x,t)\mu G(dt)). \tag{A.4}$$

But from Assumptions 1 and 2 we get

$$\begin{aligned} P(U(x,A) \le u) &= P(\text{There are no points of the process in the set} \\ &\qquad \{(t,\varepsilon):(t,\varepsilon) \in A \times R, v(x,t)+\varepsilon > u\}) \\ &= \exp(-\mu \int_{t \in A} \bar{M}(u - v(x,t))G(dt)), \end{aligned} \tag{A.5}$$

where

$$\bar{M}(y) = \int_y^{\infty} M(d\varepsilon).$$

Thus (A.4) and (A.5) imply that

$$\text{(A.6)} \qquad e^{u}\int_{t\in A}\bar{M}(u-v(x,t))G(dt) = \int_{t\in A} v^{*}(x,t)G(dt)$$

for all $A\in \mathcal{B}^{*}$.

Let x be fixed and choose

$$A = \{t\in Y: d_1 \le v(x,t) \le d_2\},$$

where $d_1 \le d_2$ are constants. Evidently $A \in \mathcal{B}^{*}$. Since $\bar{M}(y)$ is continuous and decreasing, (A.6) implies that

$$\text{(A.7)} \qquad e^{u}\bar{M}(u-d_1) \le L(d_1,d_2) \le e^{u}\bar{M}(u-d_2),$$

where

$$L(d_1,d_2) = \frac{\int_{t\in A} v^{*}(x,t)G(dt)}{G(A)}.$$

Note that $L(d_1,d_2)$ does not depend on u. Now let $d_1 \to d_2$. Then by (A.7) and the continuity property

$$\lim_{d_1\to d_2} e^{u}\bar{M}(u-d_1) = e^{u}\bar{M}(u-d_2) \le L(d_2,d_2) \le e^{u}\bar{M}(u-d_2).$$

Hence, since $L(d_2,d_2)$ is independent of u we get

$$\bar{M}(y) = L(d_2,d_2)e^{-d_2-y} = ke^{-y}$$

for some constant k. This completes the proof. Q.E.D.

LEMMA 1: *Assume Assumptions 1 and 5. Then for fixed* $p \in \bar{R}^m_+$ *and* $K \in \mathcal{B}^m_+$, *the mapping*

$$t \to \hat{v}(p,t,K) \equiv \sup_{p'x \leq f(t), x \in K} v(x,t),$$

for $t \in Y$, *is measurable.*

PROOF: Since R^m_+ is separable there exists a countable space S that is dense in $K \subset R^m_+$. As a consequence the mapping

$$t \to \sup_{p'x \leq f(t), x \in S} v(x,t)$$

is measurable. Moreover, Assumption 5 implies that for each $t \in Y$

$$\sup_{p'x \leq f(t), x \in S} v(x,t) = \hat{v}(p,t,K)$$

which proves measurability. Q.E.D.

LEMMA 2: *Assume Assumptions 1, 5 and 7. Let* $p \in \bar{R}^m_+$ *and* $K \in \mathcal{B}^m_+$ *be given where K is closed and convex. Then the mapping* $t \to \hat{x}(p,t,K)$ *defined by*

$$v(\hat{x}(p,t,K),t) = \hat{v}(p,t,K) \equiv \sup_{p'x \leq f(t), x \in K} v(x,t)$$

is measurable.

PROOF: By Lemma 7.3 in Barten and Böhm (1982), p. 397, it follows that the demand function that corresponds to the maximization problem

$$\sup_{p'x \le f(t),\, x \in K} v(x,t),$$

for given t, exists. Since the mappings $t \to v(x,t)$ and $t \to \hat{v}(p,t,K)$ are measurable (Lemma 1) it follows that $t \to \hat{x}(p,t,K)$ is measurable since

$$\hat{x}(p,t,K) = \{x \in K : v(x,t) = \hat{v}(p,t,K)\}.$$ Q.E.D.

PROOF OF THEOREM 3: By Lemma 2 the mapping $t \to \hat{v}(p, t, K)$ is well defined and measurable. Since the mapping $y \to \int_y^\infty M(d\varepsilon)$ is continuous it follows that for $z \neq z'$

$$P(\hat{v}(p,T(z),K) + E(z) = \hat{v}(p,T(z'),K) + E(z')) = 0.$$

Moreover, H a.s. contains a finite number of Poisson points and consequently there is a.s. a unique attribute, $T^*(p, C)$, that maximizes $\hat{v}(p,T(z), K) + E(z)$, s.t. $T(z) \in D$ and $(T(z), E(z)) \in H$. Then obviously the demand for the continuous commodity, $x^*(p, C)$, is given by $x^*(p, C) = \hat{x}(p,T^*(p,C), K)$ which means that $x^*(p, C)$ is unique. Q.E.D.

PROOF OF THEOREM 4: By Lemma 1, $t \to \hat{v}(p, t, K)$ is measurable. Let $A = \{T: T \leq t, T \in D \subset Y\}$. By definition:

$$G^*(t, C)$$
$$= P\Big(\sup_{T(z)\in A, (T(z),E(z))\in H, z\in Z} (\hat{v}(p,T(z),K) + E(z)) > \sup_{T(z)\in D-A, (T(z),E(z))\in H, z\in Z} (\hat{v}(p,T(z),K)$$
$$+ E(z)) \,|\, \text{There is a Poisson point in } H \cap (D\times R)\Big).$$

Since $\{T: T \leq t\}$ and $\{T: T > t\}$ are disjoint sets it follows that

$$\sup_{T(z)\in A, (T(z),E(z))\in H, z\in Z} (\hat{v}(p,T(z),K) + E(z))$$

and

$$\sup_{T(z)\in D-A, (T(z),E(z))\in H, z\in Z} (\hat{v}(p,T(z),K) + E(z))$$

are independent. Similarly to the proof of Theorem 2 it follows that for any $A \in \mathfrak{B}^*$

(A.8)
$$P\Big(\sup_{T(z)\in A, (T(z),E(z))\in H, z\in Z} (\hat{v}(p,T(z),K) + E(z)) \leq y\Big) = \begin{cases} \exp\{-e^{-y}\mu \int_A \exp(\hat{v}(p,t,K))G(dt)\} & \text{for } y \geq c, \\ 0 & \text{for } y < c. \end{cases}$$

From (A.8) we get

(A.9)
$$P\Big(\sup_{T(z)\in A, (T(z),E(z))\in H, z\in Z} (\hat{v}(p,T(z),K) + E(z)) > \sup_{T(z)\in D-A, (T(z),E(z))\in H, z\in Z} (\hat{v}(p,T(z),K)$$
$$+ E(z))\Big) = \frac{\int_{u\leq t, u\in D} \exp(\hat{v}(p,u,K))G(du)}{\int_D \exp(\hat{v}(p,u,K))G(du)} \cdot (1 - \exp(-\tilde{\Lambda}_c)),$$

where

$$\tilde{\Lambda}_c = \mu e^{-c} \int_D \exp(\hat{v}(p,u,K))G(du).$$

Since Λ_c is the expected number of Poisson points in $H \cap (D \times R)$ the probability that $H \cap (D \times R)$ is non-empty equals

(A.10) $$1 - \exp(-\tilde{\Lambda}_c).$$

Hence, the conclusion of the Theorem follows. Q.E.D.

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Footnotes

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2) A stochastic process, $\{U(x), x \in X\}$, is called a max-stable process (type III) if the following property holds:
If $\{U_i(x), x \in X\}$, $i=1,2,...,s$, are independent copies of the process, $\left\{\max_{i \le s} U_i(x), x \in X\right\}$ has the same distribution as $\{\log s + U_1(x), x \in X\}$.
The finite-dimensional distributions of a max-stable process is of the multivariate extreme value type.

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